Product Mix as a Framing Exercise:
The Role of Cost Allocation

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A short run product mix problem is used to illustrate the art of consistent framing (Demski 1994). The problem is originally represented as a linear program in the quantities of outputs and inputs. The amount of all inputs used are constrained to be less than those available. While the problem can be solved without cost allocation, it can be successfully reframed so as to preserve the optimal decision, if the allocation is performed with care. The cost allocation (reframing) exercise reduces the problem to one in outputs alone with constraints that the amount of inputs that are fixed in the short run be less than those available. Thus, the cost allocation exercise has the advantage of reducing the dimensionality of the problem in terms of the number variables and constraints. The cost allocation is non-trivial, because there exist service departments that consume each other's resources. We further demonstrate an equivalence between the cost allocation obtained using the dual of the linear program and that achieved using the standard textbook simultaneous equations (matrix, reciprocal) method performed on actual data from operations.

1. Introduction

Product costing is an art. One aspect of this art is knowing for what purpose the cost number will be used. Clark (1923) called this "different costs for different purposes." More recently, Demski (1994) has adopted a related theme: "framing." Framing refers to the description of a decision problem. How one chooses to frame a particular decision affects the calculus one should use to determine product costs. That is, what is meant by the cost of a product depends on the framework the manager invokes. "[C]ost is the very glue that connects the explicit and the implicit consumption of resources in a decision frame. Drawing the line between what is explicit and implicit at different places leaves us with different measures of cost." (Demski, 1994, p. 259)

This teaching note considers product costing in the context of a specific decision: a short run analysis of the firm's optimal product mix. We explore alternative ways of framing this problem, some of which provide (conditionally) correct answers; some do not. One approach is to ignore cost allocation, and use the full description of the technology to identify the optimal level of production. Interestingly, an accounting question arises quite naturally from this problem's dual. Consider a simplified framing where service costs are first identified. We demonstrate that there exists a way to allocate the service costs so that, within the reduced (simplified) framing, the correct product mix is preserved. That is, the cost number resulting from the accounting allocation process produces the correct implicit cost of each resource.

Factoring the implicit cost is non-trivial in the product mix decision we consider because we allow for reciprocal service departments. Reciprocal service departments not only provide their services to manufacturing departments, but to other service departments as well. It is somewhat difficult to intuit what is the implicit cost of these services. We demonstrate that the optimal product mix is obtained if and only if the service department costs are allocated to the products using the reciprocal cost allocation method.

The product mix problem, as well as the reciprocal cost allocation method, are topics that are covered in most managerial texts. The usual approach to service cost allocations begins with a planned output level for the products and input levels for the services. The texts then describe several methods for service cost allocation: direct, step and reciprocal. It is then argued, often without very much justification, that the reciprocal method is the best method for decision-making purposes.[1] The typical approach represents a bit of a paradox, as the levels of the products and services are presented exogenously. Ideally, the text would demonstrate how the allocation using the reciprocal method leads to better decisions. In this paper, we attempt to overcome this shortcoming.

The remainder of the paper is as follows. In Section 2, we describe the basic setup and frame the product mix problem two ways. The first framework involves no allocations. A large linear program is presented in which the product mix and the service department usage are choice variables. The second framework involves allocations. A smaller linear problem is presented and the following question is posed: at what rate should service department usage be charged to the products such that this program yields the same product mix as the large linear program, i.e., what rates make the two frameworks consistent? In Section 3, we show that the correct service department rates to use are equal to the shadow prices associated with service departments. And, in turn, the shadow prices are equivalent to the rates that arise if the reciprocal cost allocation method is used on even a suboptimal (but feasible) production plan. Section 4 concludes the paper.

2. The product mix problem

2. 1 Example

This subsection presents an illustrative example that will be used throughout the paper. A company produces two products, X and Y. The manufacturing process requires that the products be passed through three machines, Machines A, B, and C. Manufacturing capacity

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[1] Kaplan (1982) and Demski (1994) are notable exceptions.

(machine hours) is limited. Prices, variable costs, and manufacturing requirements per unit of each product are provided below. A linear structure with respect to revenues, costs and technology is assumed.

Parameters:

	X	Y
Selling Price	$ 50	$ 100
Direct Materials	12	38
Direct Labor	15	30
Variable Overhead (excluding service cost)	10	20
Unit contribution margin (excluding service cost)	$ 13	$ 12

Manufacturing Department Requirements	**X**	**Y**	**Capacity (Hrs.)**
Machine A	1	1	1,000
Machine B	2	1	1,500
Machine C	1	3	2,400

That is, manufacturing each unit of X requires 1 hour each on Machines A and C and 2 hours on Machine B; each unit of Y requires 1 hour each on Machines A and B and 3 hours on Machine C. The total number of available hours (the capacity) of each machine is given in the last column in the above table.

The company's operations are supported by two service departments, Power and Repair. The direct variable cost of providing services and the service requirements of the products are as follows.

Service Department Direct Variable Cost:

Power	$2.00 per kilowatt-hr.
Repair	$2.80 per repair-hr.

Service Department Requirements	**X**	**Y**	**P**	**R**
Power	1	2	0	1/2
Repair	2	1	1/3	0

That is, each unit of X requires 1 kilowatt-hr of power and 2 repair hours, and so on. Note that the service departments interact. The production of power requires repair services, and vice versa.

We now address the question of determining the optimal product mix for the firm, that is, the profit-maximizing levels of X and Y.

2. 2 Framework 1 -- calculating product mix without making allocations

In this subsection we set up a linear program (P1) in which the product mix, X and Y, and the service department usage, P and R, are viewed as choice variables and are simultaneously chosen. This program requires no allocations to be made. On the other hand, it is "large," in that there are four choice variables and it includes constraints to ensure that the demand for service department usage does not exceed the supply.

$$
\begin{aligned}
\max_{X,\,Y,\,P,\,R} \quad & 13X + 12Y - 2P - 2.8R && \text{(P1)}\\
\text{subject to:} \quad & X + Y \le 1{,}000\\
& 2X + Y \le 1{,}500\\
& X + 3Y \le 2{,}400\\
& X + 2Y + \tfrac{1}{2}R \le P\\
& 2X + Y + \tfrac{1}{3}P \le R\\
& X, Y, P, R \ge 0
\end{aligned}
$$

The objective function is the firm's total contribution margin.[2] Note the service department costs are not assigned to the products X and Y. Rather they are explicitly written out as a function of the choice variables P and R. The first three constraints recognize the limitations on manufacturing capacity. The next two constraints ensure that the number of hours required to sustain the product mix (the left-hand-side of the constraints) do not exceed the number of hours supplied by the service departments (the right-hand-side of the constraints). Finally, we impose non-negativity constraints.

A numerical optimizer (e.g., Solver in Microsoft Excel) can be used to obtain the solution to program (P1). The solution is $X^* = 300$, $Y^* = 700$, $P^* = 2{,}820$, and $R^* = 2{,}240$.

2 Firm profit differ from total contribution margin because of fixed costs. Dropping fixed costs, of course, does not affect the solution -- adding or subtracting a constant from the objective function has no impact on the optimal solution. In other words, writing the program in terms of profits or contribution margin are consistent frameworks (Demski 1992).

The company's revenues, variable costs, and total contribution margin given these levels of outputs and inputs is as follows.

Revenue	$ 85,000
Direct Materials	30,200
Direct Labor	25,500
Variable Overhead (excluding service cost)	17,000
Power (@ $2/kw-hr)	5,640
Repair (@ $2.8/repair-hr)	6,272
Total Contribution Margin	$ 388

2. 3 Framework 2 -- allocating costs and then calculating product mix

Is it possible to write a program that yields the same product mix as (P1) but in which P and R are not choice variables? The smaller two-variable program will yield the correct product mix if the unit contribution margin of each product (net of service costs) is correctly computed. But what rates should be used to charge products for services?

Let r_P denote the rate at which the products are charged for using 1 kilowatt-hr of power. Similarly, let r_R denote the rate for repair department usage. Hence, the unit contribution margin of product X and Y are $13 - 1r_P - 2r_R$ and $12 - 2r_p - 1r_R$, respectively. The product mix problem (P2) is presented below. The goal is to choose r_R and r_P such that the optimal product mix under (P2) is the same as under (P1), i.e., X = 300 and Y = 700 is the solution to (P2).

$$\max_{X,\,Y} \quad (13 - 1r_P - 2r_R)\,X + (12 - 2r_p - 1r_R)\,Y \qquad \text{(P2)}$$

$$\begin{aligned}
\text{subject to:}\quad X + Y &\le 1{,}000\\
2X + Y &\le 1{,}500\\
X + 3Y &\le 2{,}400\\
X, Y &\ge 0
\end{aligned}$$

3. Computing the correct charge for service usage

3.1 Ignoring service department interaction

The reader may be tempted to try to use the direct variable cost of services when solving (P2), i.e., $r_P = 2$ and $r_P = 2.8$. This results in unit contribution margins for X and Y of 5.4 and 5.2, respectively. The solution to (P2) using these unit contribution numbers is $X^* = Y^* = 500$. The levels of the services necessary to support these output quantities can be derived from the equations that ensures demand for services equals their supply.

$$\begin{aligned} X + 2Y + \tfrac{1}{2}R &= P \\ 2X + Y + \tfrac{1}{3}P &= R \end{aligned}$$

The optimal levels of the services are $P^* = 2{,}700$ and $R^* = 2{,}400$. The company's revenues, variable costs, and contribution margin given these levels of outputs and inputs is as follows.

Revenue	$ 75,000
Direct Materials	25,000
Direct Labor	22,500
Variable Overhead (excluding service cost)	15,000
Power	5,400
Repair	6,720
Total Contribution Margin	$ 380

Note the company incurs an opportunity cost of 388 - 380 = 8 when the problem is framed under (P2) and direct variable costs are used as rates. The reason is that using the direct variable costs to allocate costs ignores the indirect costs of services attributable to their interdependent nature. To see this note that the total service costs are 5,400 + 6,720 = 12,120 while the costs that are allocated to products X and Y are 500[1(2)+2(2.8)] + 500[2(2)+1(2.8)] = 7,200. Thus, the implied cost allocation is not "tidy." The unallocated costs, 12,120 - 7,200 = 4,920, equal the indirect service costs, $2(\tfrac{1}{2}R) + 2.8(\tfrac{1}{3}P) = 4{,}920$.

A necessary condition for an allocation to be valid is that it be tidy (Thomas, 1974).[3] Of course, that still leaves many allocation schemes to choose from. We next try to determine the allocation scheme that is not only tidy but that also yields the optimal product mix.

3.2 Inferring rates from the solution

In this subsection, we use the solution to (P1) to infer the correct rates. Of course, this defeats the purpose of framing the exercise as (P2). It is not useful to frame the product mix problem as a small linear program if the input to this program can only be obtained by solving the larger program! In the next two subsections, we show how the rates (and the correct contribution margins) in (P2) can be derived without solving (P1). But for now, let us indulge in the backward logic.

At the optimal solution, the contribution margin of product X and Y are $300(13 - 1r_P - 2r_R)$ and $700(12 - 2r_p - 1r_R)$, respectively, and the power and repair costs are \$5400 and \$6,720. One may be tempted to take power costs assigned to product X and Y and equate it to power costs incurred as follows: $300(1r_P) + 700(2r_P) = 5{,}400$. A similar equation can be setup for repair costs: $300(2r_R) + 700(2r_R) = 6{,}720$. The two equations imply $r_P = \frac{5400}{1700} = 3\ \frac{3}{17}$ and $r_R = \frac{6720}{1300} = 5\,\frac{11}{65}$. Substituting these values for r_P and r_R and X = 300 and Y = 700 into the objective function in (P2) yields the total contribution margin of 388. But we still have not asked the important question: Will using these rates and solving (P2) yield X = 300 and Y = 700 as the optimal solution? The answer is no. Substituting $r_P = 3\ \frac{3}{17}$ and $r_R = 5\,\frac{11}{65}$ into the objective function in (P1) and solving produces an incorrect product mix of X = 0 and Y = 800.

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[3] Thomas (1974) specifies three criteria that make an allocation method theoretically justifiable. First, the allocation method should be tidy: the allocated part should sum to the whole. Second, the allocation method should be unambiguous: it must be unique given an objective function. Third, the allocation method should be defensible: there must exist an underlying logic for the objective function that is employed. This note provides a setting in which the reciprocal cost allocation method satisfies all three criteria.

The problem with the above approach is that service department costs are allocated taking into account only the direct usage of services by the products. The fact that service departments are using each others services in no way affected the two equations we used to solve for the rates. This resulted in incorrect inference of unit contribution margins. The correct procedure is to solve the equations that set the total contribution margin equal to 388 and total service costs equal to 5,400 + 6,720 = 12,120, as shown below.

$$300(13 - 1r_P - 2r_R) + 700(12 - 2r_p - 1r_R) = 388$$
$$300(1r_P + 2r_R) + 700(2r_p + 1r_R) = 12{,}120$$

Solving the two simultaneous equations yields $r_P = 3.52$ and $r_R = 4.56$. Using these rates, the unit contribution margin for X and Y are 0.36 and 0.40, respectively. Hence, (P2) is:

$$\begin{array}{llcccl}
\max\limits_{X,\,Y} & 0.36\,X + 0.4\,Y & & & & \\
\text{subject to:} & X & + & Y & \leq & 1{,}000 \\
 & 2\,X & + & Y & \leq & 1{,}500 \\
 & X & + & 3\,Y & \leq & 2{,}400 \\
 & X,\,Y & & & \geq & 0
\end{array}$$

The solution to this program is X = 300 and Y = 700, the same as under (P1). This is not a coincidence. By equating the total service cost and total contribution margin to that obtained under (P1) we have, by mere construct, ensured that the correct contribution margins are used in (P2). The key is to use the correct rates. The question is whether these rates can be obtained *without* having to solve (P1). The answer is yes. In the next subsection, we show these rates are the shadow prices of the service constraints in (P1) and, hence, fall right out of the dual of the large program.

3.3 Using shadow prices

Most numerical optimizers, in addition to providing the solution to the linear program, also give you the shadow prices (dual variables). The reader can verify that the shadow prices for the five constraints in (P1) are 0.34, 0, 0.02, 3.52, and 4.56, respectively. Note the shadow prices on the service department constraints are just the cost allocation rates we are looking for. Hence, we now know where to go to find the correct rates: to the dual of (P1).

Every linear program is associated with a unique dual program and the solution to the dual program yields the shadow prices. Denote the dual variables of the five constraints in (P1) by w_1 through w_5. For (P1), the dual is denoted by (D1) and is as follows.

$$\min_{w_i} \quad 1{,}000\, w_1 + 1{,}500\, w_2 + 2{,}400\, w_3 \tag{D1}$$

$$\begin{array}{llcl} \text{subject to:} & w_1 + 2w_2 + w_3 + w_4 + 2w_5 & \geq & 13 \\ & w_1 + w_2 + 3w_3 + 2w_4 + w_5 & \geq & 12 \\ & -w_4 + \frac{1}{3} w_5 & \geq & -2 \\ & -\frac{1}{2} w_4 - w_5 & \geq & -2.8 \\ & w_1, w_2, w_3, w_4, w_5 & \geq & 0 \end{array}$$

The dual variables associated with the constraints in the dual program are the choice variables in (P1). That is, the dual variables associated with the constraints in (D1) are X, Y, P, and R, respectively. Programs (P1) and (D1) are duals of each other.

At the optimal solution, we know that P and R are positive (at least some services are needed). This implies that the corresponding dual constraints (the third and the fourth constraints in (D1)) hold as equalities. These constraints can be rearranged and written as follows:

$$w_4 = 2 + \frac{1}{3} w_5 \tag{1}$$

$$w_5 = 2.8 + \frac{1}{2} w_4 \tag{2}$$

The solution of the above simultaneous equations is $w_4 = 3.52$ and $w_5 = 4.56$. Why these equations give the correct allocation rates is easily seen. Equation (1) states that the cost used to allocate 1 kilowatt of power is the direct cost ($2) plus the indirect cost of $\frac{1}{3}$ repair hours that are needed to sustain the 1 kilowatt of power ($\frac{1}{3}$ w_5). Similarly, equation (2) sums the direct and indirect costs associated with the repair department. It is these indirect costs that were ignored earlier. The dual program permits us to get the correct rates without having to solve (P1) or, for that matter, even (D1). All we needed to do was to solve a set of linear equalities. Using these cost allocation rates makes (P1) and (P2) consistent frameworks. The reader can operate in either framework without having to worry about any distortions in product mix.

The criteria for choosing an allocation method in this setting is now apparent. We would like to choose an allocation method which results in the correct rates to charge products for the services they use and, hence, the correct unit contribution margins for each product. In the next subsection, we show that reciprocal method meets this test.

3.4 Using reciprocal cost allocation method

Choose any arbitrary feasible level of X, Y, P and R under which, for each service, demand equals supply. That is, pick any X, Y, P, and R that satisfies all the constraints in (P1) and under which the two service department constraints hold as equalities. One such solution is X = 250, Y = 250, P = 1,350, R = 1,200. Continuing, one would calculate the reciprocal cost allocations by first organizing the data as follows (this is the format in which the data is most often presented in the textbook chapter on service cost allocation).

Service Dept. Usage	**P**	**R**	**X**	**Y**	**Total**
Power (in kilowatt-hrs)	0	600	250	500	1,350
Repair (in repair-hrs)	450	0	500	250	1,200

		Proportion of costs allocated to:				
Service Dept.	**Direct Cost**	**P**	**R**	**X**	**Y**	**Total**
Power	2(1350) = \$2,700	0	$\frac{12}{27}$	$\frac{5}{27}$	$\frac{10}{27}$	1
Repair	2.8(1350) = \$3,360	$\frac{9}{24}$	0	$\frac{10}{24}$	$\frac{5}{24}$	1

Denote the total cost (sum of direct and indirect) associated with power and repair departments by C_P and C_R, respectively. The reciprocal equations are:

Total costs = Direct costs in service dept. + Costs allocated to service dept.

$$C_P = 2700 + \frac{9}{24} \quad C_R$$

$$C_R = 3360 + \frac{12}{27} \quad C_P$$

Solving for C_P and C_R yields C_P = 4,752, C_R = 5,472. The allocations are provided below.

		Allocated to:	
Service Dept.	**Total Cost**	**X**	**Y**
Power	\$4,752	$\frac{5}{27}(4752)$	$\frac{10}{27}(4752)$
Repair	\$5,472	$\frac{10}{24}(5472)$	$\frac{5}{24}(5472)$

The unit contribution margin of X is $13 - \dfrac{[\frac{5}{27}(4752) + \frac{10}{24}(5472)]}{250} = 0.36.$

The unit contribution margin of Y is $12 - \dfrac{[\frac{10}{27}(4752) + \frac{5}{24}(5472)]}{250} = 0.40.$

The rate being charges for services provided by the power department is $\dfrac{C_P}{1350} = \dfrac{4752}{1350} =$ 3.52.

The rate being charges for services provided by the repair department is $\dfrac{C_R}{1200} = \dfrac{5472}{1200} = 4.56.$

Sure enough, the reciprocal method gives the correct rates and correct unit contribution margins. Hence, (P1) can be equivalently framed as (P2) as long as services are allocated at rates determined using the reciprocal cost method. The reader can verify that the other commonly used alternatives, direct or step methods, do not give the correct rates.

4. Conclusion

In this paper, we frame the product mix problem with interacting service departments two ways. The first way involves no cost allocations. The second way uses cost allocations, and in this framework it is of course important to allocate these costs so as to preserve the no cost allocation solution. This exercise is interesting because of the interacting nature of the service departments. The decision-maker has to be cognizant of the fact that a product that uses direct services only from one department, say electric power, also utilizes the services of other departments that support the power department. The resultant implicit costs of these services decrease the contribution margin of the final product.

The first framework is large, and takes the interdependence of the services into account because it explicitly recognizes the demand and supply constraints for the services at the same time as the produce mix is chosen. The second framework is smaller, and implicitly recognizes the interaction through an initial stage of cost allocation. Within this second framework, it was important to allocate these costs in such a way as to produce a "correct" contribution margin. This places demands on the cost allocation besides tidiness.

The cost allocation can be accomplished several ways. First, the rates can determined by solving simultaneous equations which are intuitive. These are equivalent to the dual constraints on the services in the large framework. Second, and somewhat interesting, the traditional textbook method of performing reciprocal service allocation, which does not take an optimization perspective, can also be used. As a practical matter, one needs only start with a feasible production plan, implement the textbook procedure, and the appropriate cost allocation can be directly derived. This lends support for textbook statements that offer the reciprocal method as the preferred one.

Cost allocation is an exercise in disaggregation: taking a whole and breaking it up into parts. Therefore, by definition, there is a level of arbitrariness associated with allocations. The arbitrariness of allocations has been the subject of several debates. The worry is that arbitrary allocations are difficult to defend, and without a defensible theory of allocations, allocations undermine the validity of the numbers on the financial statements (see, for example, Thomas, AAA, 1974, p. xiii). This note provides a setting in which the reciprocal cost allocation method is defensible.

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