

A MODEL OF INTERNATIONAL ASSET PRICING*

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In this paper an intertemporal model of international asset pricing is constructed which admits differences in consumption opportunity sets across countries. It is shown that the real expected excess return on a risky asset is proportional to the covariance of the return of that asset with changes in the world real consumption rate. (World real consumption does not, in general, correspond to a basket of commodities consumed by all investors.) The model has no barriers to international investment, but it is compatible with empirical facts which contradict the predictions of earlier models and which seem to imply that asset markets are internationally segmented.

1. Introduction

Without a model showing how assets are priced in a world in which asset markets are fully integrated, it is impossible to determine whether asset markets are segmented internationally or not.¹ Many issues in financial economics cannot be dealt with unless an assumption is made about whether markets are segmented internationally or not. For instance, the widespread use, in all countries which have a stock market, of some proxy of the home-country market portfolio to test how home-country assets are priced can be justified only by an assumption that markets are internationally segmented. As another example, only if markets are fully integrated is it true that projects with perfectly correlated cash-flows are valued in the same way regardless of the country in which they are undertaken. As a final example, it is only when markets are fully integrated that it is always optimal, in a mean-variance framework, to diversify internationally.

Presently, there does not appear to exist conclusive evidence showing that

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¹In this paper, asset markets are said to be perfectly integrated internationally if two assets (existing or hypothetical) which have perfectly correlated returns in a given currency but belong to different countries have identical expected returns in that currency. Markets are said to be segmented if this condition does not hold.

the hypothesis of segmented national stock markets can be rejected.² It might well be the case that markets are indeed segmented.³ One can, however, argue that the reason the hypothesis of market segmentation cannot be rejected is that the models used to test it offer an insufficient description of how assets are priced in a world in which markets are fully integrated and exchange rates are flexible.⁴ Existing models of international asset pricing proceed from one of two assumptions; the world can be modeled as if there is only one commodity,⁵ while countries have different rates of inflation, or it can be assumed that the terms of trade (i.e., the relative price of imports) are perfectly correlated with the exchange rate.⁶ Such assumptions are unsatisfactory, as it is known that the terms of trade change stochastically through time and are not perfectly correlated with exchange rates. Those empirical facts about exchange rate dynamics imply that at each point in time, the consumption opportunity set of an investor depends on the country in which he resides.⁷ The consumption opportunity set of an investor is defined here as the set of goods available for his consumption, the current prices and the distribution of the future prices of those goods. The fact that consumption opportunity sets differ today does not mean that investors believe they will differ forever. A vast body of empirical literature indicates that whenever consumption opportunity sets differ, international trade in commodities and international factor mobility will, over time, reduce differences between consumption opportunity sets. Existing models of international asset pricing, however, assume that consumption opportunity sets are the same across countries at any point in time.

Earlier models of international asset pricing make predictions which are not compatible with observed facts. Contrary to the prediction of those models, it does not seem to be the case that an asset whose suitably defined excess return has a covariance equal to zero with the return on some world

²Aliber (1978) and Solnik (1977) review the literature. Rodriguez and Carter (1979) summarize the evidence and conclude their discussion by implying that markets are segmented (see p. 498).

³Models with barriers to international investment are offered by Adler and Dumas (1975, 1976), Black (1979), Kouri (1976) and Stulz (1981).

⁴The model developed in this paper does not hold if exchange rates are fixed, except for adjustments at discrete intervals of time.

⁵See, for instance, Grauer, Litzberger and Stehle (1976), Kouri (1978) and Fama and Farber (1979). In the following, the terminology of Holmes (1967) is used and the assumption that the world can be modeled as if there exists only one good is called the assumption of 'naive' purchasing power parity.

⁶See, for instance, Solnik (1974).

⁷There is a large literature on exchange rate dynamics. For a review of the literature on purchasing power parity, see Officer (1976) and Katseli-Papaefstratiou (1979). Genberg (1978) and the other papers in the same issue of the *Journal of International Economics* give empirical evidence on purchasing power parity during the seventies. Dornbusch (1978), Bilson (1979) and Isard (1978), among others, review current ideas on exchange rate dynamics.

market portfolio, will have an expected excess return equal to zero.⁸ It has been argued that assets which have the same covariance with the world market portfolio have different excess returns if they belong to different home countries.⁹ Finally, investors hold a much larger amount of risky assets of their home country than predicted by those models.¹⁰

One could claim that the fact that those predictions of earlier models of international asset pricing do not hold shows that markets are segmented internationally. In the present paper, a model of international asset pricing is developed on the assumption that markets are fully integrated. It includes earlier models as special cases. The model can hold even when the predictions of earlier models are contradicted by observed facts.

In the model developed in this paper, consumption opportunity sets can differ across countries. Section 2 describes the investment and consumption opportunity set of investors. Section 3 discusses the asset demands of investors and argues that investors with identical utility functions living in different countries can have different asset demands when consumption opportunity sets differ across countries. Section 4 derives the fundamental asset pricing equation of this paper, while section 5 discusses the economic implications of that equation. Section 5 shows that predictions of earlier models do not necessarily hold in a world in which markets are fully integrated but investors have different consumption opportunity sets. The same section also derives implications of the model for the risk premium incorporated in the forward exchange rate. Section 6 offers concluding remarks.

2. Consumption and investment opportunity sets

2.1. Consumption opportunity sets

It is assumed that a domestic investor can buy K different commodities. The path of the price $P(i)$ of the i th domestic commodity is described by a stochastic differential equation written as¹¹

$$dP(i)/P(i) = \mu_{P(i)}(s, t)dt + \sigma_{P(i)}(s, t)dz_{P(i)}, \quad i = 1, \dots, k, \quad (1)$$

where s is an $S \times 1$ vector of state variables, $dz_{P(i)}$ is a Wiener process, $\mu_{P(i)}$ is the instantaneous mean and $\sigma_{P(i)}^2$ is the instantaneous variance of the

⁸See Stehle (1978).

⁹See references in Aliber (1978).

¹⁰Black (1978, p. 8) writes: 'What we have to understand is not why foreign investment occurs, but why it isn't much more common.'

¹¹See Merton (1971) for an introduction to those equations. Additional references are given in Breeden (1979).

percentage rate of change of the price of the i th good. By assumption the state variables follow Ito processes. An intuitive way to understand eq. (1) is to look at it as a reduced-form equation for the rate of change of the price of commodities. The vector of state variables is understood to include all the state variables which affect at least one investor's expected intertemporal utility. Without loss of generality, it is assumed that there are two countries, and asterisks are used to designate prices of commodities and assets in the foreign country. (K^* commodities are available in the foreign country.)

The first K state variables in the vector s are taken to be the logarithms of the prices of the commodities available in the domestic country. The vector s^* is defined as the vector of state variables rearranged so that the first K^* state variables are the logarithms of the prices in foreign currency of the commodities available in the foreign country. Some of the other state variables which could belong in the vector s are specified later.

The functional relationships between the state variables and the instantaneous mean and variance of the rate of change of commodity prices do not need to be further restricted for the purpose of this paper. Markets are assumed to be always in equilibrium. The only important restriction on the path of the price of good i given by eq. (2) is that, in non-mathematical terms, it must be smooth. This restriction is important because it prevents our model from being used to study a world in which exchange rates are fixed, except for adjustments at discrete time intervals.¹²

Let the domestic price of one unit of foreign currency at time t be e_t . If there are no obstacles to international commodity arbitrage, in the sense that arbitrage can always be made instantaneously at zero costs, then for identical traded commodities, the law of one price holds exactly, i.e.,

$$P_t(j) = e_t P_t^*(j), \quad \forall j \in G_t, \quad (2)$$

where $P_t^*(j)$ is the price in foreign currency of good j and G_t is the set of traded commodities at time t . The set of traded goods does not need to be constant through time, but it is assumed that the set of consumed goods in each country is constant. In this paper, it is always assumed that the law of one price, as stated in eq. (2), holds for all traded commodities.

There exists no theoretical or empirical reason to assume that eq. (2) holds all the time for all commodities.¹³ For simplicity, a commodity for which the law of one price, as stated in eq. (2), does not hold at time t is viewed as a non-traded commodity at time t . If a commodity is non-traded at time t , this does not mean that that commodity will be non-traded forever. In fact, for

¹²Kouri (1976) has a model in which that restriction does not hold.

¹³See Kravis and Lipsey (1978) and Isard (1977) for good empirical evidence. Samuelson (1948) was among the first authors to stress the role of non-traded commodities.

most commodities, today's expectation of how the prices of those commodities will deviate from some version of the law of one price at some future date, not too close from today, is likely to be equal to zero.¹⁴ A large number of studies have shown that short-run deviations from the law of one price, as stated in eq. (2), are not trivial.

It is assumed that the exchange rate follows an equation similar to the equation which describes the price dynamics of the i th domestic commodity,

$$de/e = \mu_e(s, t)dt + \sigma_e(s, t)dz_e, \quad (3)$$

where μ_e is the instantaneous mean and σ_e^2 is the instantaneous variance of the percentage rate of change of the exchange rate. The exchange rate dynamics, together with the price dynamics and the law of one price, yield the dynamics for the foreign currency price of the i th traded commodity, $P^*(i)$. Differentiate (2) using Ito's Lemma to get the dynamics of $P^*(i)$,

$$\frac{dP^*(i)}{P^*(i)} = -\frac{de}{e} + \frac{dP(j)}{P(j)} - \frac{de}{e} \frac{dP(j)}{P(j)} + \left(\frac{de}{e}\right)^2, \quad (4)$$

where $(de/e)dP(j)/P(j)$ is equal to the instantaneous covariance between the percentage rate of change of the exchange rate and the percentage rate of change of the domestic currency price of the j th commodity, and $(de/e)^2$ is equal to the instantaneous variance of the percentage rate of change of the exchange rate.

In a world with only one commodity, eq. (4) simply states that 'naive' purchasing power parity holds.¹⁵ In the present paper, differences in tastes and consumption opportunity sets across countries make it impossible, in general, to model the world as if there is only one commodity.

2.2. *Investment opportunity sets*

All assets are assumed to be traded, i.e., the domestic price of a foreign asset is the price in foreign currency of that asset multiplied by the exchange rate. The return in domestic currency of a foreign asset is different from the return in foreign currency of the same asset because the return in domestic currency depends both on the change in the foreign currency price of the asset and on the change in the exchange rate. It is assumed that: (1) for each asset the returns accrue only in the form of capital gains; (2) there are no

¹⁴See Roll (1979) for the view that purchasing power parity (PPP) holds *ex ante* for some relevant period of time.

¹⁵'Naive' purchasing power parity is that version of PPP which states that a change in the domestic price level is immediately offset by a change in the exchange rate. See Holmes (1967).

transaction costs; (3) unlimited short-sales with full use of the proceeds are permitted; and (4) markets are always in equilibrium.

If the i th foreign asset is risky in the sense that its instantaneous return is stochastic, it is assumed that its price $I^*(i)$ follows a stochastic differential equation,

$$\frac{dI^*(i)}{I^*(i)} = \mu_{I^*(i)}(s, t)dt + \sigma_{I^*(i)}(s, t)dz_{I^*(i)}. \quad (5)$$

Eq. (5) gives the instantaneous rate of return in foreign currency for the i th foreign asset. Let $D(i)$ be the domestic price of the i th foreign asset, i.e., $eI^*(i)$. The dynamics of $D(i)$ are obtained by using Ito's Lemma,

$$\frac{dD(i)}{D(i)} = \frac{dI^*(i)}{I^*(i)} + \frac{de}{e} + \frac{de}{e} \frac{dI^*(i)}{I^*(i)}. \quad (6)$$

Inspection of eq. (6) shows that the return for a domestic investor on a foreign risky asset is different from the return on that asset for a foreign investor, because the exchange rate changes through time.

It is assumed that there are N securities which have risky nominal returns in both countries, and n of those securities are domestic securities. In each country, there is one nominal bond. Let $I(j)$ be the price in domestic money of a domestic risky asset. $I(j)$ follows a stochastic differential equation of the same form as the one followed by $I^*(i)$. Let B (B^*) be the price in domestic (foreign) currency of the domestic (foreign) nominal safe asset. The dynamics for B are given by

$$dB/B = R(s, t)dt, \quad (7)$$

where R is simply the instantaneous nominal rate of return on a safe domestic bond. R^* is the instantaneous nominal rate of return on the foreign default-free bond whose foreign currency price is given by B^* .

3. Properties of asset demand functions

3.1. Definition of excess returns

Define the instantaneous excess return of the i th foreign asset for a domestic investor as the instantaneous return of an investment of one unit of domestic currency in the i th foreign asset financed by borrowing abroad at the interest rate R^* . The instantaneous excess return of the i th foreign asset

for a domestic investor, written $dH(i)/H(i)$, is

$$\frac{dH(i)}{H(i)} = \frac{dI^*(i)}{I^*(i)} + \text{cov}(d \ln I^*(i), d \ln e) - R^* dt. \quad (8)$$

The instantaneous excess return on the i th foreign asset for a domestic investor is perfectly correlated with the instantaneous return of that asset in foreign currency. The stochastic part of the change in the exchange rate (i.e., $\sigma_e dz_e$) does not enter the instantaneous excess return of a foreign asset and, consequently, the excess return on holdings of foreign assets is hedged against unanticipated changes in the exchange rate. To keep his holdings of foreign assets hedged through time, the investor must continuously adjust his borrowings so that his net investment in foreign risky assets is always equal to zero.

In the following, the return on an investment of one unit of domestic (foreign) currency in a risky asset i by a domestic (foreign) investor, financed by borrowing in the home-country of that asset, is called the excess return for a domestic (foreign) investor on the i th asset. For a foreign investor, the instantaneous excess return on the j th domestic asset is

$$\frac{dH^*(j)}{H^*(j)} = \frac{dI(j)}{I(j)} - \text{cov}(d \ln I(j), d \ln e) - R dt. \quad (9)$$

For all investors in a country, the excess return on each asset is the same. Because of the covariance terms in eqs. (8) and (9), the excess return for a domestic investor on a domestic asset is, in general, different from the excess return on the same asset for a foreign investor.

The excess return on the foreign safe nominal bond for a domestic investor is defined as the return on one unit of domestic currency invested in the foreign bond, financed by borrowing at the interest rate R in the domestic country, i.e.,

$$\frac{dH(B)}{H(B)} = R^* dt + \frac{de}{e} - R dt. \quad (10)$$

This equation implies that the excess return on a foreign safe nominal bond is perfectly correlated with the change in the exchange rate. For a foreign investor, the return on a domestic safe nominal bond is risky and its instantaneous excess return is given by

$$\frac{dH^*(B)}{H^*(B)} = R dt + \text{var}(d \ln e) - \frac{de}{e} - R^* dt. \quad (11)$$

Let $\boldsymbol{\mu}$ be the $(N+1) \times 1$ vector of expected excess returns for domestic investors. The N -first elements of $\boldsymbol{\mu}$ comprise the expected excess returns on assets which are risky in both countries, whereas the last element of $\boldsymbol{\mu}$ is the expected excess return on a foreign bond. In the following, a vector is designated by a bold italic lower-case letter, whereas a matrix is designated by a bold roman capital letter. $\boldsymbol{\mu}^*$ is the $(N+1) \times 1$ vector of expected excess returns for foreign investors. The last element of $\boldsymbol{\mu}^*$ is the expected excess return on a domestic bond for foreign investors. The N -first elements of $\boldsymbol{\mu}$ and $\boldsymbol{\mu}^*$ correspond each to the expected excess return of identical assets which are held by investors belonging to different countries. The n domestic risky assets are the first n assets in $\boldsymbol{\mu}$ and $\boldsymbol{\mu}^*$.

3.2. Asset demands

Let $[\mathbf{w}^k; b^k]$ be the investments in risky assets of investor k , expressed as a fraction of his wealth, where \mathbf{w}^k is an $1 \times N$ vector whose representative element w_i^k is the proportion of his wealth investor k invests in the i th asset, which is an asset risky in both countries. Let $b^k W^k$ be the investor's investment in foreign bonds in excess of his holdings of foreign bonds required to hedge his investments in foreign risky assets against exchange rate risks.¹⁶ Finally, define \mathbf{V}_{aa} ($\mathbf{V}_{a^*a^*}$) as the $(N+1) \times (N+1)$ variance-covariance matrix of excess returns for domestic (foreign) investors and \mathbf{V}_{as} ($\mathbf{V}_{a^*s^*}$) as the covariance matrix of those excess returns with changes in the state variables in \mathbf{s} (\mathbf{s}^*). Assuming that each investor maximizes a well-behaved, state-independent von Neuman–Morgenstern expected utility function of lifetime consumption, the optimization problem of the investor is formally equivalent to the problem faced by an investor in Breeden (1979).¹⁷ The optimal portfolio of domestic investor k is characterized by the following investments in risky assets:

$$[\mathbf{w}^k; b^k]' = \left(\frac{T^k}{C_w^k W^k} \right) \mathbf{V}_{aa}^{-1} \boldsymbol{\mu} + \mathbf{V}_{aa}^{-1} \mathbf{V}_{as} \left(\frac{-C_s^k}{C_w^k W^k} + \left\{ \frac{C^k \boldsymbol{\alpha}^k}{C_w^k W^k} - \frac{\mathbf{m}^k T^k}{C_w^k W^k} \right\} \right), \tag{12}$$

where $C^k(W, \mathbf{s}, t)$ is the consumption expenditure function of investor k and $U^k(C(W, \mathbf{s}, t), \mathbf{P}, t)$ is the investor's indirect utility function of consumption

¹⁶To hedge his investments which are risky in the foreign country, the investor borrows abroad at the interest rate R^* an amount equal to $h^k = \sum_{i=n+1}^N w_i^k W^k$. To get the investor's total holdings of foreign bonds, one has to subtract h^k from $b^k W^k$. When investors hedge using bonds, $b^k W^k$ is the net foreign investment of domestic investor k .

¹⁷See Breeden (1979, sect. 7).

expenditures when the prices of the available commodities are given by the vector \mathbf{P} . The partial derivatives of the consumption expenditure function with respect to the state variables form the $S \times 1$ vector \mathbf{C}_s^k , whereas C_w^k is the partial derivative of the consumption expenditure function with respect to wealth. By convention, C^k is equal to $\sum_{i=1}^k P_i c_i^k$, where c_i^k is the number of units of commodity i consumed by investor k . $\boldsymbol{\alpha}^k$ is the $K \times 1$ vector of average expenditure shares of the K goods available in the domestic country, i.e., $P_i c_i^k / C^k = \alpha_i^k$. \mathbf{m}^k is the $K \times 1$ vector of marginal expenditure shares of the same goods, i.e., $P_i \partial c_i^k / \partial C^k = m_i^k$. Finally, T^k is the absolute risk tolerance of the investor, i.e., $-U_c^k / U_{cc}^k$, whereas $\mathbf{0}$ is an $(S - K) \times 1$ vector of zeros.

Let $[w^j; b^j]$ be the investments in risky assets of investor j , who is a foreign investor, expressed as a fraction of his wealth. Whereas w^j is the vector of fractions of the wealth W^j of the investor invested in assets which are risky in both countries, b^j is the fraction of the wealth of the investor invested in domestic safe bonds, in addition to his holdings of domestic safe bonds required to hedge his investments in risky domestic assets against foreign exchange risks. W^j is the wealth of the foreign investor in his home currency. The optimal portfolio of investor j is characterized by

$$\begin{aligned}
 [w^j; b^j]' &= \left(\frac{T^j}{C_w^j W^j} \right) [\mathbf{V}_{a^* a^*}]^{-1} \boldsymbol{\mu}^* + [\mathbf{V}_{a^* a^*}]^{-1} \mathbf{V}_{a^* s^*} \\
 &\times \left(\frac{-C_{s^*}^j}{C_w^j W^j} + \left\{ \frac{C^j \boldsymbol{\alpha}^j}{C_w^j W^j} - \frac{\mathbf{m}^j T^j}{C_w^j W^j} \right\} \right), \tag{13}
 \end{aligned}$$

where $C^j(W, s^*, t)$ is the consumption function of investor j in his home currency, $\boldsymbol{\alpha}^j$ is his $K^* \times 1$ vector of average expenditure shares and \mathbf{m}^j his $K^* \times 1$ vector of marginal expenditure shares on the K^* goods available in his country.

3.3. Comparison of asset demands

The domestic investor is indifferent between holding a linear combination of $S + 1$ mutual fund portfolios and his portfolio of risky assets.¹⁸ One mutual fund portfolio is the portfolio which corresponds to the point of tangency of the capital market line and the efficient frontier of portfolios of

¹⁸Merton is the first to have mentioned this result, for instance Merton (1973); formal proofs of the result and discussions of the properties of the mutual funds can be found in Breeden (1979), Richard (1979) and Merton (1981). To check the result, notice that (12) can be rewritten as a weighted sum of column vectors, where the column vectors do not depend on the preferences of investor k , whereas the weights do.

risky assets in the space of nominal expected returns and standard deviations of those returns for domestic investors. That portfolio will be called the tangency portfolio for domestic investors and written $[\mathbf{w} : b]$. The S other mutual fund portfolios are portfolios which, among all feasible portfolios, have the highest possible correlation with the S state variables. The $S+1$ mutual funds result also applies to foreign investors.

Note, however, that a mutual fund which is positively correlated with the exchange rate in the domestic country (i.e., e) is negatively correlated with the exchange rate in the foreign country (i.e., $1/e$). If a mutual fund in the domestic country has an investment in an asset perfectly positively correlated with the domestic exchange rate, it is long in the foreign safe bond. Abroad, a mutual fund which holds an asset perfectly positively correlated with the exchange rate in the domestic country is short in the domestic safe bond. It follows that the composition of the $S+1$ mutual fund portfolios held by foreign investors is (at least for some portfolios) different from the composition of the $S+1$ mutual fund portfolios held by domestic investors. The tangency portfolio for foreign investors is written $[\mathbf{w}^* : b^*]$.

The relationship between the tangency portfolio for domestic investors and the tangency portfolio for foreign investors is given in the following proposition:

Proposition 1. The tangency portfolio held by domestic investors is different from the tangency portfolio held by foreign investors. However, the proportion in which two assets risky in both countries are held by domestic investors is the same as the proportion in which they are held by foreign investors.

Proof. See appendix.

Proposition 1 places no restrictions on the menu of assets available to domestic and foreign investors. One asset can be a foreign nominal bond with maturity at date t' , such that $t' > t + dt$. If a domestic investor holds such a bond to maturity, his return in domestic currency over the life of the bond will be perfectly correlated with the total change in the exchange rate over the life of the bond. Such a bond plays no special role in a mean-variance efficient portfolio. Notice that Proposition 1 does not depend on assumptions about the nature of the relationship between exchange rate changes and changes in commodity prices. Whether naive purchasing power parity holds or not has no impact on the composition of the mean-variance efficient mutual fund portfolio obtained by using nominal excess returns.

All domestic investors invest in the same 'hedge' mutual funds, i.e., the S mutual funds most highly correlated with state variables in the vector \mathbf{s} . In

the setting of this paper, two investors who have the same intertemporal utility function and identical wealth in domestic currency will invest in different 'hedge' mutual funds if they reside in different countries. This result is explained by the fact that consumption investment opportunity sets differ across countries. Suppose, for instance, that the dollar price of a haircut in Rome differs from the dollar price of a haircut in New York. Italian investors who want to hedge against unanticipated changes in the price of haircuts do not, in this example, hold the same portfolio as American investors who want to hedge against unanticipated changes in the price of haircuts, even if the utility function of all investors is the same. Differences in consumption opportunity sets *can* create meaningful differences in asset holdings across countries.

4. The asset pricing equation

In this section, an asset pricing relationship is obtained for a world of flexible exchange rates in which tastes and consumption opportunity sets differ across countries.

4.1. Aggregation within countries

Define $C(i)$ as the consumption expenditures of the i th domestic investor, i.e., $C(i) = C^i(W, s, t)$. Using Breeden (1979), eq. (12) can be transformed into

$$\mu - V_{as} \begin{pmatrix} m^i \\ \theta \end{pmatrix} = (T^i)^{-1} \left\{ V_{aC(i)} - V_{as} C(i) \begin{pmatrix} \alpha^i \\ \theta \end{pmatrix} \right\}, \tag{14}$$

where $V_{aC(i)}$ is the vector of covariances of home-country returns of risky assets with changes in consumption expenditures of investor i .

Let $Pm(i)$ be the price of a basket of commodities which contains $m_j^i P_k / m_k^i P_j$ units of commodity j , for all j 's, where m_j^i is the marginal expenditure share of commodity j for the i th investor. Define $P\alpha(i)$ as the price of a basket of commodities which contains $\alpha_j^i P_k / \alpha_k^i P_j$ units of commodity j , for all j 's, where α_j^i is the average expenditure share of commodity j for investor i . $Pm(i)$ ($P\alpha(i)$) is the price of a basket of commodities normalized so that it contains exactly one unit of commodity k and so that the expenditure on each commodity in the basket is equal to $m_j^i Pm(i)$ ($\alpha_j^i P\alpha(i)$). No generality is lost by assuming that there exists a traded commodity k which is consumed by all investors. As the first K state variables are the logarithms of the prices of the commodities available to a domestic investor, eq. (14) is equivalent to

$$\mu - V_{aPm(i)} = (T^i)^{-1} \{ V_{aC(i)} - C(i) V_{aP\alpha(i)} \}, \tag{15}$$

where $V_{aP\alpha(i)}$ is the $(N+1) \times 1$ vector of covariances of excess returns for domestic investors with $d \ln P\alpha(i)$, and $V_{aPm(i)}$ is the vector of covariances of excess returns with $d \ln Pm(i)$.

Adding eq. (15) across all domestic investors yields

$$\mu - V_{aPm(D)} = (T^D)^{-1} \{ V_{aC(D)} - V_{aP\alpha(D)} C(D) \}, \tag{16}$$

where, if \sum_D indicates the operation of summing across all domestic investors, $T^D = \sum_D T^i$, $C(D) = \sum_D C(i)$, $P\alpha(D) = \sum_D (C(i)/C(D)) P\alpha(i)$, and $Pm(D) = \sum_D (T^i/T^D) Pm(i)$. $P\alpha(D)$ is the price of a basket of commodities which contains $\alpha_j(D) P_k / \alpha_k(D) P_j$ units of the j th commodity, where $\alpha_j(D)$, for all j 's, is the fraction of domestic consumption expenditure spent on the j th commodity. $Pm(D)$ is the price of a basket of commodities which contains $m_i(D) P_k / m_k(D) P_i$ units of the j th commodity, such that $m_i(D)$ is a risk tolerance weighted average of the marginal expenditure share of the j th commodity for all domestic investors.

$1/Pm(D)$ is a locally accurate measure of the real value of a marginal increase in domestic consumption expenditure due to a change in the value of the investment of domestic investors in their tangency portfolio. The asset demand functions imply that the increase in consumption expenditures of an investor i due to a change in the value of the tangency portfolio is equal to the ratio of the risk-tolerances of investor i and j , i.e., T^i/T^j , times the increase in consumption expenditures of investor j due to the same cause. Suppose $T^i/T^j = 2$ and $C^j_w W^j = C^i_w W^i$. In this case, investor i invested twice as much as investor j in the tangency portfolio, and an unanticipated increase in the value of the tangency portfolio increases his consumption expenditures twice as much as it increases the consumption expenditures of investor j . The basket of commodities which is bought if consumption expenditures increase in the domestic country due to an increase in the value of the tangency portfolio reflects the fact that the increase in consumption expenditures of investor i is twice as large as the increase in consumption expenditures of investor j .

An equation similar to eq. (16) is obtained for the foreign country,

$$\mu^* - V_{a^*P^*m(F)} = (T^{*F})^{-1} \{ V_{a^*C^*(F)} - V_{a^*P^*\alpha(F)} C^*(F) \}. \tag{17}$$

The asterisks in (17) indicate that everything is computed from the perspective of foreign investors who use their home currency. For instance, μ^* stands for the vector of expected excess returns for foreign investors and $P^*\alpha(F)$ stands for the price in foreign currency of a basket of commodities which contains $\alpha_j(F) P^*(k) / \alpha_k(F) P^*(j)$ units of commodity j , where $\alpha_j(F)$ is the fraction of foreign aggregate consumption spent on the j th foreign

commodity. The capital letter F indicates that eq. (17) is obtained by summing asset demand equations across all foreign investors.

Because nominal variables in eq. (16) are in domestic currency and excess returns are viewed from the perspective of a domestic investor, whereas nominal variables in (17) are in foreign currency and viewed from the perspective of a foreign investor, it is not possible to obtain a meaningful relationship for expected excess returns by simply summing eqs. (16) and (17).

4.2. *Aggregation across countries*

If the equation which describes equilibrium expected returns for foreign investors is multiplied by the exchange rate, it contains foreign aggregate consumption in domestic currency, i.e.,

$$\mu^* - V_{a^*P^*m(F)} = (T^F)^{-1} \{eV_{a^*C^*(F)} - V_{a^*P^*z(F)}C(F)\}, \tag{18}$$

where $C(F)$ is foreign consumption expenditure in domestic currency and $T^F = eT^{*F}$. $C(F)$ is equal to $eC^*(F)$, which implies that $V_{a^*C^*(F)}$, i.e., the vector of covariances of excess returns with changes in foreign aggregate consumption in foreign currency, is equivalent to $V_{a^*(1/e)C(F)}$. Using Ito's Lemma, $V_{a^*(1/e)C(F)}$ is equal to

$$V_{a^*(1/e)C(F)} = \frac{1}{e} (V_{a^*C(F)} - V_{a^*e}C(F)), \tag{19}$$

where V_{a^*e} is the vector of covariances between home currency returns and $d \ln e$. The same transformation, applied respectively to $C(F)V_{a^*P^*z(F)}$ and $T^F V_{a^*P^*m(F)}$, yields

$$C(F)V_{a^*P^*z(F)} = (C(F)V_{a^*Pz(F)} - C(F)V_{a^*e}), \tag{20}$$

$$T^F V_{a^*P^*m(F)} = T^F (V_{a^*Pm(F)} - V_{a^*e}). \tag{21}$$

In the domestic country, it is not possible, in general, to buy the basket of commodities whose price abroad is $P^*\alpha(F)$ at the domestic currency price $P\alpha(F) = eP^*\alpha(F)$. In that sense $P\alpha(F)$ is not observable in the domestic country and neither is $Pm(F) = eP^*m(F)$. In principle, $P^*m(F)$ and $P^*\alpha(F)$ are observable in the foreign country.

Note that the excess return abroad of a security which is not a bond with a safe instantaneous return differs from the excess return at home of the same security by a non-stochastic term [see, for instance, eq. (9)]. The excess

return abroad on the domestic bond which has a safe instantaneous return is equal to the excess return at home of the foreign bond with a safe instantaneous return multiplied by minus one plus a non-stochastic term [see eqs. (10) and (11)]. Using these facts, it can easily be shown that

$$\mathbf{L}\boldsymbol{\mu}^* = \boldsymbol{\mu} - \mathbf{V}_{a^*e}, \quad (22)$$

where \mathbf{L} is a diagonal matrix with ones everywhere along the diagonal except for minus one in the last row. Furthermore, if x is a stochastic variable,

$$\mathbf{V}_{ax} = \mathbf{L}\mathbf{V}_{a^*x}. \quad (23)$$

Premultiply (18) by $T^F\mathbf{L}$ to get

$$T^F(\mathbf{L}\boldsymbol{\mu}^* - \mathbf{L}\mathbf{V}_{a^*P^*m(F)}) = \mathbf{L}e\mathbf{V}_{a^*C^*(F)} - \mathbf{L}\mathbf{V}_{a^*P^*z(F)}C(F). \quad (24)$$

Substitute into (24) eqs. (19) through (22) and use the result given by (23) to obtain

$$\boldsymbol{\mu} - \mathbf{V}_{aPm(F)} = (T^F)^{-1} \{ \mathbf{V}_{aC(F)} - C(F)\mathbf{V}_{aP\alpha(F)} \}. \quad (25)$$

Summing eqs. (16) and (25) yields

$$\boldsymbol{\mu} - \mathbf{V}_{aPm} = (T^W)^{-1} \{ \mathbf{V}_{aC} - \mathbf{V}_{aPz}C \} \quad \text{where} \quad T^W = T^F + T^D. \quad (26)$$

Here $\boldsymbol{\mu}$ is the vector of expected excess returns for domestic investors. The excess return of an asset for a domestic investor is defined as the return in the domestic country of a portfolio with a long holding of that asset financed by selling the nominal safe bond of the home-country of the asset. T^W is a measure of world risk-tolerance. \mathbf{V}_{aC} is the vector of covariances between home-country returns of risky assets and changes in world consumption in domestic currency. \mathbf{V}_{aPm} and $\mathbf{V}_{aP\alpha}$ are the vectors of covariances of home-country returns with $d \ln Pm$ and $d \ln P\alpha$. An increase in Pm means a decrease in the real value of a marginal dollar of consumption expenditures, whereas an increase in $P\alpha$ means a fall in the real value of total consumption expenditures. Eq. (26) is the fundamental pricing equation of this paper.

The left-hand side of eq. (26) can be interpreted as the vector of expected excess *real* returns of risky assets. An unanticipated increase in the value of the investors' tangency portfolios induces an increase in aggregate consumption expenditures which is distributed among investors in proportion to their risk tolerances.

Each investor spends his increase in consumption expenditures on a basket of commodities defined by his vector of marginal expenditure shares. The

risk tolerance weighted average of marginal expenditure shares for a commodity gives the fraction of the increase in world consumption expenditures spent on that commodity when those expenditures increase because of an increase in the value of the tangency portfolios. (Remember that the tangency portfolio of domestic investors is different from the tangency portfolio of foreign investors.)

The risk tolerance weighted average of the vectors of marginal expenditure shares yields a commodity basket whose price is Pm . By construction, therefore, the basket of commodities whose price is Pm is the basket of commodities bought by the world as a whole when world consumption expenditure increases due to an increase in the value of the tangency portfolios. Applying Ito's Lemma, it immediately follows that the left-hand side of the asset pricing equation corresponds to the vector of expected excess returns of risky assets measured in units of the basket of commodities with price Pm .

The vector in curly brackets on the right-hand side of the asset pricing equation is easier to interpret. $1/P\alpha$ is a locally accurate measure of the real value of one dollar of consumption expenditures spent in exactly the same way as world consumption expenditures are spent. Let c be real consumption, i.e., $C/P\alpha$. It is easy to verify that $P\alpha V_{ac} = V_{ac} - C V_{aP\alpha}$, which is the term in curly brackets in eq. (26). The following proposition must now hold:

Proposition 2. The expected excess real return of a risky asset is proportional to the covariance of the home currency return of that asset with changes in world real consumption rate.

Two facts are the key to understanding Proposition 2. First, the higher the covariance of an asset's payoffs with marginal utility of consumption expenditure, the more valuable that asset is. A continuous function of a variable which satisfies a stochastic differential equation of the Ito-type is perfectly correlated with that variable. This fact makes it possible to state that the expected real return an investor requires on a risky asset is proportional to its covariance with changes in the real consumption expenditure of the investor. In equilibrium, however, what is relevant for the pricing of an asset is how much that asset is correlated with the common non-diversifiable part of the changes in consumption expenditure of all investors.

The second important fact is that, because investors can form a portfolio of common stocks whose return does not depend directly on unanticipated changes in the exchange rate, they will not be rewarded for bearing the exchange rate risks. If a domestic investor holds a foreign stock hedged against exchange rate risks, the return on that stock is perfectly correlated with the return on the same common stock for a foreign investor.

5. Economic implications

5.1. *The pricing of common stocks*

Earlier models of international asset pricing yield the result that suitably defined expected excess returns on common stocks are proportional to the covariance of those returns with the return on some suitably defined market portfolio.¹⁹ The predictions of the model presented here can be made equivalent to the predictions of earlier models by making additional assumptions which imply that changes in world real consumption are perfectly correlated with the return on some world market portfolio. The world market portfolios implied by various earlier models are so highly correlated that it is not clear that those models are empirically distinguishable.²⁰

The main use of those earlier models of international asset pricing has been to test the hypothesis that asset markets are segmented internationally. The segmentation hypothesis means that two assets which belong to different countries but have the same risk with respect to some model of international asset pricing without barriers to international investment have different expected excess returns. No study has successfully rejected the hypothesis that markets are segmented using earlier models of international asset pricing. If changes in world real consumption are not perfectly correlated with the return on some world market portfolio used by earlier studies, the measure of risk for assets used here is not equivalent to the measure used in earlier studies. This implies that it is possible for a test of the segmentation hypothesis which uses an earlier model of international asset pricing to fail to reject the hypothesis when in fact there are no barriers to international investment and expected returns satisfy the asset pricing equation developed in this paper. Since the revealed preference of investors for home-country assets is compatible either with the existence of barriers to international investment or with the fully integrated markets model presented here, more sophisticated tests of the segmentation hypothesis are required.

It should now be clear that one has to be careful about capital budgeting in open economies. If earlier asset pricing models hold, it does not matter where a firm produces. A scale-expanding project would not change the discount rate of a firm even if the scale-expanding project is located in a foreign country. With the model presented here, the location of a project matters if that project uses non-traded goods as inputs or produces non-traded goods. The discount rate of a project, in general, depends on where it is located even in a world without barriers to international investment.

¹⁹See references given in section 1.

²⁰See Solnik (1978).

5.2. The pricing of bonds

Let $f(t, dt)$ be the forward exchange rate for a contract expiring at date $t+dt$ agreed upon at time t . Define $\ln \{f(t, dt)/e(t)\} = \mu_f dt$, where $e(t)$ is the exchange rate today. The interest rate parity theorem²¹ states that if R is the rate of return on a domestic safe nominal bond and if R^* is the rate of return on a safe foreign nominal bond,

$$\mu_f = R - R^*. \quad (27)$$

Ceteris paribus, the lower μ_e , i.e., the expected rate of change of the exchange rate, the higher the forward exchange rate with respect to the expected future spot exchange rate. If the expected future spot exchange rate is higher than the forward exchange rate, the expected return on investing in a foreign nominal bond is higher uncovered than covered (i.e., hedged against exchange rate risks). It follows that the investor must pay a risk premium to hedge his investment against exchange rate risks. Using the asset pricing model developed in this paper, the relationship between μ_f and μ_e is given by

$$R - R^* = \mu_f = \mu_e - V_{ePm} - (\beta_{ec}/\beta_{Mc})[\mu_M - V_{MPm}], \quad (28)$$

where β_{Mc} is the covariance of changes in real consumption with the return of portfolio M , which can be any observable portfolio, μ_M is the expected excess return of that portfolio, and V_{MPm} is the covariance of the return of that portfolio with $d \ln Pm$, where Pm is the price of a commodity basket identified earlier.

The key point of eq. (28) is stated in the following proposition:

Proposition 3. Ceteris paribus, an increase in the covariance between changes in the exchange rate and changes in the world real consumption rate decreases the forward exchange rate.

The risk premium incorporated in the forward exchange rate, as given by eq. (28), can be different from zero when other models imply it should be zero and vice versa.²² For instance, the empirical fact that the exchange rate is correlated with the terms of trade can be sufficient, by itself, to create a risk premium. It has been argued that if safe nominal bonds belong to the

²¹See, for instance, Officer and Willet (1970).

²²Discussions of the risk premium incorporated in the forward exchange rate can be found, for instance, in Solnik (1974), Roll and Solnik (1977), Kouri (1976, 1978), Fama and Farber (1979), Frankel (1979) and Grauer, Litzenberger and Stehle (1976). For good empirical evidence that the forward exchange rate is not an unbiased predictor of the future spot exchange rate, see, for instance, Hansen and Hodrick (1980).

market portfolio, i.e., if there are 'outside' nominal assets like government bonds, there is a risk premium. The argument can be stated in this way: if some assets in the market portfolio have real returns negatively correlated with the rate of change of the price level, the exchange rate will, in general, have a non-zero beta, since it will be correlated with those assets. Safe nominal bonds have real returns perfectly negatively correlated with the rate of change of the price level. This fact is then interpreted to mean that a positive supply of so-called 'outside' assets is a sufficient condition for the existence of a risk premium.²³ (Note however, that there exist the theoretical possibility that the effect on beta of the correlation of the exchange rate with the real value of other assets offsets the effect on beta of its correlation with the real return of nominal bonds.) In the model presented in this paper, a change in the supply of government bonds does not necessarily affect the risk premium incorporated in the forward exchange rate. Many papers have been devoted to the fact that a change in the supply of so-called 'outside' assets does not leave the budget constraint of the government unchanged.²⁴ Unless one specifies exactly how the government budget constraint is affected by a change in the supply of outside assets, it is not possible to say, on theoretical grounds, how a change in the supply of 'outside' assets affects the risk premium incorporated in the forward exchange rate.

The generality of eq. (26) needs to be stressed. Earlier models which yield a formula for the risk premium incorporated in the forward exchange rate require a constant expected rate of change for the exchange rate (or, alternatively, that no asset has a return correlated with the expected rate of change of the exchange rate). Such an assumption leads to models which cannot account for the empirical fact that the risk premium incorporated in the forward exchange rate seems to change through time. It must also be noted that the asset pricing equation which leads to eq. (24) can be used to price the forward exchange rate for forward contracts for all maturities.

5.3. *Correlation of consumption across countries*

Assume that, for all prices, all investors consume some positive quantity of each good available in their home country. With this assumption, the following result holds:

Proposition 4. If (a) markets are complete (in the sense that an unconstrained Pareto-optimal equilibrium is achieved) and (b) consumption opportunity sets differ across countries, then changes in the real consumption rates are not perfectly correlated across countries.

²³See Frenkel (1979) and also Kouri (1977) and Fama and Farber (1979).

²⁴See, for instance, Barro (1974).

To prove Proposition 4, choose the $(K + K^* - G)$ first state variables to be the logarithms of the prices in domestic currency of commodities available in at least one country. Let all vectors of average and marginal expenditure shares have dimension $(K + K^* - G) \times 1$ and \mathbf{V}_{ss} be the $S \times S$ variance-covariance matrix of state variables. $\boldsymbol{\mu}_s$ is the vector of excess expected returns on assets perfectly correlated with state variables. It is shown in the appendix that the covariance between changes in the consumption of the i th domestic investor and changes in the domestic currency consumption of the j th foreign investor is

$$\begin{aligned} \text{cov}(C^i, eC^i) &= e \left[T^i \boldsymbol{\mu}_s - \mathbf{V}_{ss} \begin{pmatrix} \mathbf{m}^i \\ \boldsymbol{\theta} \end{pmatrix} T^i \right]' \mathbf{V}_{ss}^{-1} \left[T^i \boldsymbol{\mu}^j - \mathbf{V}_{ss} \begin{pmatrix} \mathbf{m}^j \\ \boldsymbol{\theta} \end{pmatrix} T^i \right] \\ &\quad + eC^i \begin{pmatrix} \boldsymbol{\alpha}^i \\ \boldsymbol{\theta} \end{pmatrix}' \left[T^i \boldsymbol{\mu}_s - \mathbf{V}_{ss} \begin{pmatrix} \mathbf{m}^i \\ \boldsymbol{\theta} \end{pmatrix} T^i \right] \\ &\quad + \left[T^i \boldsymbol{\mu}_s - \mathbf{V}_{ss} \begin{pmatrix} \mathbf{m}^i \\ \boldsymbol{\theta} \end{pmatrix} T^i \right]' eC^i \begin{pmatrix} \boldsymbol{\alpha}^i \\ \boldsymbol{\theta} \end{pmatrix} + C^i \begin{pmatrix} \boldsymbol{\alpha}^i \\ \boldsymbol{\theta} \end{pmatrix}' \mathbf{V}_{ss} \begin{pmatrix} \boldsymbol{\alpha}^i \\ \boldsymbol{\theta} \end{pmatrix} eC^i. \end{aligned} \quad (29)$$

Changes in the domestic currency consumption of investor i and investor j can be perfectly correlated only if $\text{cov}(C^i, eC^i)$ is equal to the product of the standard deviations of changes in C^i and eC^i . It is easy — but cumbersome — to show that this happens *only* if the investors are the same (except for their risk tolerance if utility functions are homothetic) and consumption opportunity sets do not differ. It follows that only if 'naive' purchasing power parity holds are changes in real consumption perfectly correlated across countries. This result is important because it allows one to understand the assumption required to price domestic assets using only domestic data, i.e., the assumption of 'naive' purchasing power parity. If this assumption is not correct, domestic risky assets will be priced differently if they are correlated with foreign consumption than if they are not.

5.4. Empirical research

The analysis conducted so far suggests some new directions for empirical research in international finance. The most important question which emerges from this paper is: are the real expected excess returns of risky assets proportional to their covariances with changes in world real consumption? Some of the problems associated with testing this hypothesis are briefly discussed in this section.

One important advantage of the approach developed here is that consumption data are easily available for most countries, whereas no data are available on the value of invested wealth in most countries.

Unfortunately, consumption data are never computed in a way which is ideally suited for economic analysis. For instance, consumption data nearly always include purchases of durable goods, rather than the value of the services provided by the existing stock of durable goods.

The empirical relevance of international asset pricing considerations depends here on the correlation of changes in real consumption in various countries with changes in world real consumption. This suggests that a useful preliminary test would be to look at whether or not changes in U.S. real consumption have a correlation coefficient with changes in world real consumption that is statistically different from one. If this correlation coefficient is not statistically different from one, using U.S. data on consumption when measuring the risk of an asset could be an acceptable procedure. Furthermore, such a result would indicate that differences in consumption opportunity sets are not likely to matter very much for studies of international asset pricing.

If changes in U.S. real consumption are not too highly correlated with changes in world real consumption, it is theoretically possible to test the hypothesis that the risk of a U.S. asset is measured by the covariance of its return with changes in world real consumption rather than with changes in U.S. real consumption. The problem which arises in practice with such a test is that consumption betas are not likely to be stable over time. If those non-stationarities turn out to be important, the full vector of state variables must be identified and observable to test the hypothesis. In this case, the informational requirements of a test of the asset pricing equation are not smaller than those of a test of an international version of Merton's multi-beta asset pricing equation.²⁵

It is well-known that investors hold portfolios heavily weighted towards assets of their home country. One explanation advanced for that fact is that there are barriers to international investment. In section 3 of this paper, it is argued that another possible explanation is that home country assets form better hedges against state variables which affect the intertemporal expected utility of investors of a given country. It is possible to examine empirically whether a portfolio highly correlated with changes in the real consumption of a given country contains a relatively large proportion of assets of that country. Notice, however, that state variables which are relevant for investors in a particular country need not affect expected excess returns significantly.

6. Concluding remarks

In this paper, a model of international asset pricing which admits differences — albeit temporary — in consumption opportunity sets has been

²⁵See Merton (1973). Cornell (1981) makes this point with respect to Breeden (1979). Richard (1979) has discussed the conditions under which a multi-beta model has stationary betas.

constructed. It has been shown that such a model yields a simple asset pricing equation, which states that the real expected return of a risky asset is proportional to the covariance of the home country return of that asset with changes in world real consumption rate. The model contains earlier models of international asset pricing as special cases and is compatible with some empirical facts which seem to contradict the predictions of earlier models of international asset pricing.

While this paper does not model differences in consumption opportunity sets across countries, it seems that an interesting extension of the present work would be to look at a more simple model in which differences in consumption opportunity sets would be studied explicitly. For instance, a model with transportation costs which are a decreasing function of time would generate interesting differences in consumption opportunity sets and would introduce an explicit role for the current account (i.e., net foreign investment) in a model of international asset pricing.

The present model does not take money into account explicitly. However, the nature of the process followed by the money supply in both countries is likely to affect the risk premium incorporated in the forward exchange rate. A possible extension of the present work would be to construct an explicit model of the money market equilibrium in both countries.

Appendix

A.1. Proof of Proposition 1

Let λ be a scalar, \mathbf{V}_{II} the $N \times N$ variance-covariance matrix of excess returns of assets which are not bonds with instantaneous maturity, \mathbf{V}_{Ie} the $N \times 1$ vector of covariances of the excess returns of these risky assets with the exchange rate, V_{ee} the variance of the exchange rate, $\boldsymbol{\mu}^I$ the $N \times 1$ vector of expected excess returns on these risky assets. Using the definition of \mathbf{w} and \mathbf{w}^* ,

$$\mathbf{w} = \lambda (\mathbf{V}_{II} - \mathbf{V}'_{eI} \mathbf{V}_{II}^{-1} \mathbf{V}_{Ie})^{-1} [\boldsymbol{\mu}^I - \mathbf{V}_{Ie} V_{ee}^{-1} (R^* + \mu_e - R)], \tag{A.1}$$

$$\mathbf{w}^* = \lambda^* (\mathbf{V}_{I^*I^*} - \mathbf{V}'_{eI^*} (\mathbf{V}_{I^*I^*})^{-1} \mathbf{V}_{I^*e})^{-1} [\boldsymbol{\mu}^{I^*} + \mathbf{V}_{I^*e} V_{ee}^{-1} (R + \sigma_e^2 - \mu_e - R^*)]. \tag{A.2}$$

Asterisks used as superscripts in (A.2) indicate that excess returns for foreign investors are used.

First, the terms in square brackets are compared. Let \mathbf{I}^J be an $N \times N$ diagonal matrix with zeros everywhere except for ones in the last $N - n$ diagonal elements. \mathbf{I} is an $N \times N$ identity matrix. $\boldsymbol{\mu}^I$ is the $N \times 1$ vector of

home-currency excess returns of risky assets which are not bonds with instantaneous maturity. μ is an $N \times 1$ vector which has R in its first n elements and R^* everywhere else. Then

$$\begin{aligned} & [\mu^{I^*} + V_{I^*e} V_{ee}^{-1} (R + \sigma_e^2 - \mu_e - R^*)] \\ &= \mu^I - (I - I^f) V_{I^*e} - \mu - V_{I^*e} V_{ee}^{-1} (R^* - \sigma_e^2 + \mu_e - R) \\ &= \mu^I + I^f V_{I^*e} - \mu - V_{I^*e} V_{ee}^{-1} (R^* + \mu_e - R). \end{aligned} \tag{A.3}$$

If $V_{I^*e} = V_{Ie}$, the last line of (A.3) is equal to the term in square brackets of (A.1). A typical element of V_{I^*e} is

$$E \left\{ \left(\frac{dI_i}{I_i} - \frac{de}{e} \frac{dI_i}{I_i} - R dt \right) \frac{de}{e} \right\} = E \left\{ \frac{de}{e} \frac{dI_i}{I_i} \right\}, \quad i \leq n. \tag{A.4}$$

It immediately follows that $V_{I^*e} = V_{Ie}$. A typical element of V_{II} is equal to the product of two terms of the same form as the term in parentheses in (A.4). The term in parentheses depends on the residence of the investor only because of the covariance term, which vanishes in products. It follows that $V_{II} = V_{I^*I^*}$. This means that the terms in the first parentheses of (A.1) are equal to those of the first parentheses of (A.2), which completes the proof.

A.2. Proof that eq. (27) holds

Proposition 4 holds only if markets are complete. If markets are complete, it is possible to construct portfolios which are perfectly correlated with the S state variables. Let V_{ss} be the $(S + 1) \times (S + 1)$ variance-covariance matrix of the returns of those S portfolios and the market portfolio, whereas w_s^i is the $(S + 1) \times 1$ vector of proportions of wealth invested in those $S + 1$ portfolios by investor i . Using Ito's Lemma and the assumption of complete markets [as in Breeden (1979)], it follows that

$$\begin{aligned} \text{cov}(eC^i, C^i) &= C_w^i w_s^{i'} V_{ss} w_s^i W^i W^i e C_w^i + e C_w^i w_s^{i'} V_{ss} C_s^i W^i \\ &+ e C_w^j w_s^{j'} V_{ss} C_s^j W^j + C_s^i V_{ss} C_s^j e. \end{aligned} \tag{A.5}$$

To obtain $\text{var}(C^i)$ let $e = 1$ in (A.5), whereas to obtain $\text{var}(eC^j)$, let $C^i = eC^j$ in (A.5). $\text{Cov}(eC^j, C^i)$ is the sum of four terms, where each term can be written in terms of the distribution of asset returns and the utility function of investors by substituting out w_s^i and w_s^j in (A.5) from the asset demand functions. Let μ_s be the vector of expected excess returns on the $S + 1$

portfolios. The asset demands for investor i can be written, using (12), as

$$\mathbf{w}_i^j = \left(\frac{T^j}{C_w^j W^j} \right) \mathbf{V}_{ss}^{-1} \boldsymbol{\mu}_s + \left(\frac{-C_s^j}{C_w^j W^j} + \left\{ \frac{C^j \boldsymbol{\alpha}^j}{C_w^j W^j} \frac{\mathbf{m}^j T^j}{C_w^j W^j} \right\} \right) \quad (\text{A.6})$$

Using (A.6), the first term on the right-hand side of (A.5) can be rewritten as

$$\begin{aligned} & C_w^i \mathbf{w}_s^{i'} \mathbf{V}_{ss} \mathbf{V}_{ss}^{-1} \mathbf{V}_{ss} \mathbf{w}_s^j e C_w^j W^j W^j \\ &= \left[T^i \boldsymbol{\mu}_s - \mathbf{V}_{ss} C_s^i + \mathbf{V}_{ss} C^i \left(\begin{matrix} \boldsymbol{\alpha}^i \\ \boldsymbol{\theta} \end{matrix} \right) - \mathbf{V}_{ss} \left(\begin{matrix} \mathbf{m}^i \\ \boldsymbol{\theta} \end{matrix} \right) T^i \right] \\ & \quad \times \mathbf{V}_{ss}^{-1} \left[e T^j \boldsymbol{\mu}_s^* - \mathbf{V}_{ss} e C_s^j + \mathbf{V}_{ss} e C^j \left(\begin{matrix} \boldsymbol{\alpha}^j \\ \boldsymbol{\theta} \end{matrix} \right) - \mathbf{V}_{ss} \left(\begin{matrix} \mathbf{m}^j \\ \boldsymbol{\theta} \end{matrix} \right) e T^j \right]. \end{aligned} \quad (\text{A.7})$$

Using (A.6), the second term on the right-hand side of (A.5) can be rewritten as

$$C_s^i \mathbf{V}_{ss} \mathbf{w}_s^j e C_w^j = C_s^{i'} \left[e T^j \mathbf{w}_s^j - \mathbf{V}_{ss} e C_s^j + \mathbf{V}_{ss} e C^j \left(\begin{matrix} \boldsymbol{\alpha}^j \\ \boldsymbol{\theta} \end{matrix} \right) - \mathbf{V}_{ss} \left(\begin{matrix} \mathbf{m}^j \\ \boldsymbol{\theta} \end{matrix} \right) T^j e \right]. \quad (\text{A.8})$$

The third term on the right-hand side of (A.5) can be obtained from (A.8) by letting investor i be the foreign investor and investor j be the domestic investor. This implies that all terms in square brackets in (A.8) are in this case expressed in domestic currency, whereas C_s^i is replaced by $e C_s^i$.

From (A.5), (A.7) and (A.8) it is straightforward to obtain eq. (27).

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