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Feedback and the Success of Irrational Investors

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Feedback and the Success of Irrational Investors

We provide a model in which irrational investors trade based upon considerations that have no inherent connection to fundamentals. However, trading activity affects market prices, and because of feedback from security prices to cash flows, the irrational trades influence underlying cash flows. As a result, irrational investors can, in some situations, earn positive expected profits. These expected profits are not market compensation for bearing risk, and can exceed the expected profits of rational informed investors. Although the trading of irrational investors cause prices to deviate from fundamental values, stock prices follow a random walk.

1 Introduction

Investors often share common misconceptions, and participate in common errors of analysis. For example, a substantial number of investors employ technical rules that are supported by neither conceptual considerations nor empirical evidence. Fads of investment in industry sectors, methods of security analysis, and simplistic theories of the stock market tend to proliferate through the mass media and word of mouth (Shiller (2000) discusses such phenomena). Groups of investors who have fallen prey to common elementary errors, such as confusing the company Telecommunications Incorporated with the firm with ticker symbol TCI, have caused large price movements in one stock based upon news arrival in another unrelated stock (see Rashes (2001)). As another indication of the commonality of trading errors, investors and prices sometimes react to the republication of information that is already public (see Ho and Michaely (1988) and Huberman and Regev (2001)).

Anecdotally, during the late 1990s it became increasingly popular to value stocks based upon ad hoc heuristics. For example, many analysts (and presumably the investors who listened to them) began to value tech firms based upon revenue rather than earnings; and to value e-commerce firms based upon eyeballs rather than revenue. It has been alleged that these valuation methods were inappropriate, a criticism that, at least with the benefit of hindsight, seems to carry some weight.[1] With the rise of the internet, there has been increased opportunity for investors to gain improved information about stocks. However, it is also easier for questionable stock market theories to be spread rapidly and widely.

There is a growing literature that explores the effects of irrational trading on market prices, and the profitability of such trading.[2] While our paper contributes to this literature, our focus is very different. In contrast to the existing literature, irrational trading in our model does not provide profit opportunities to uninformed rational investors,

[1] See, for example, "Eyeballs, Bah! Figuring Dot-Coms' Real Worth," *Business Week Online*, October 30, 2000, or www.fvginternational.com/industries/industries_internet.html, that respectively discuss the invalidity and validity of these approaches. Ofek and Richardson (2003) provide evidence suggesting that the internet bubble was driven by the naïvely optimistic trading of some investors

[2] Hirshleifer (2001) and Barberis and Thaler (2003) review this literature.

even if they are aware that there are traders with psychological biases in the market. Indeed, from the perspective of the rational, but uninformed, investors, the market is informationally efficient. Nevertheless, irrational trading affects prices, and thereby affects firms' fundamental values. Moreover, the irrational investors in our model can, under some conditions, earn positive expected profits that can even exceed the profits of rational informed traders.

We are not the first to consider conditions under which irrational traders can earn higher expected profits than fully rational ones. However, in previous work, the irrational traders either earn high average profits by having greater exposure to systematic risk, or achieve higher average risk-adjusted profits by more aggressively exploiting private information.[3,4] In our setting prices are set by risk neutral market makers, so that the expected profits of irrational investors in our model do not derive from exposure to priced risk. Moreover, the irrational investors in our model have no "inherent" private information relating to the cash flows that firms generate. However, the misperceptions of irrational investors affect these underlying cash flows because of feedback from stock prices to cash flows. As a result, in equilibrium, irrational trading is correlated with cash flows.

Feedback, which plays a central role in our model, can arise in practice for a variety of reasons. For example, a higher stock price may help firms attract customers and

[3]In DeLong, Shleifer, Summers, and Waldmann (1991), investors with fundamental information underestimate risk, and therefore take larger long positions in the risky asset. Therefore these investors take fuller advantage of the asset's risk premium than their rational counterparts. Thus, in DeLong et al, high irrational returns reflect a premium for market risk. Several other papers have examined how in an imperfectly competitive securities market overconfident informed traders can benefit by trading more aggressively on private information. [In an imperfectly competitive securities market, such aggressiveness can intimidate other informed traders; see Kyle and Wang (1997), Wang (1998) and Fischer and Verrecchia (1999). In a competitive securities market, Hirshleifer and Luo (2001) show that overconfident investors who trade aggressively in response to their private information signals can exploit liquidity traders more profitably than rational investors.] All of these papers, however, require that irrational investors have direct information about fundamentals.

[4]Another strand of literature goes beyond the analysis of trading profitability to consider long-run consumption/wealth accumulation when individuals make intertemporal consumption/investment decisions. In Blume and Easley (1990) and Kogan, Ross, Wang, and Westerfield (2003), irrational investors who have utility functions that are closer to logarithmic than those of rational investors can accumulate more wealth in the long-run than their rational counterparts. Kogan et al also show that irrational traders can sometimes have a long-term price impact even if their relative wealth converges to zero over time.

employees, may reduce the firm's cost of capital, and may provide a cheap currency for making acquisitions.[5] Higher stock prices also can encourage increased investment in complementary technologies. Several recent papers have provided evidence suggesting that irrational mispricing feeds back into corporate behavior (Baker, Stein, and Wurgler (2003), Polk and Sapienza (2003), and Dong, Hirshleifer, Richardson, and Teoh (2003)). In our model, feedback takes the form of stakeholders in the firm (such as suppliers, customers, and employees) being more willing to make firm-specific investments, such as increased effort, if they have more favorable expectations about the firm's prospects.[6]

Another central feature of our model is that sentiment causes different irrational investors to trade at different times. Some irrational investors may be infused with sentiment before others. Alternatively, some may be faster than others in mobilizing to submit trades. Furthermore, as in many behavioral finance models, we assume that rational investors do not directly observe the irrational trading.[7] Some rational investors receive private information about future cash flows, and some are uninformed.

Within this setting, if irrational investors are bullish, prices tend to increase first when the early irrationals buy, and then again when the late irrationals buy. Irrational investors earn positive expected profits when they trade early but earn negative expected profits when they trade late. In the absence of feedback, because of the market impact of their trades, the average profit of the irrational investor is negative. However, with feed-

[5] Subrahmanyam and Titman (2001) discuss feedback in a related context. Feedback of the type examined in this paper is also related to what Soros (2003) refers to as his "theory of reflexivity," which holds that stock prices affect fundamentals.

[6] Anecdotally, feedback from stock prices to the behavior of potential stakeholders was evident during the dot-com bubble of the 1990s. MBA students from top programs such as Stanford and Wharton were leaving school without their degrees to pursue employment with new dot-coms, spurning once coveted investment banking positions (see *Fortune*, "MBAs Get Dot.com Fever," (8/2/99)). After stock prices collapsed, so did the willingness of MBAs to work for dot-coms (see, e.g., *Duke Magazine* May-June (2003)). Similarly, high level executives were lured from secure positions at bricks-and-mortar firms for the promise implicit in dot-com firms with high stock prices.

[7] For example, in DeLong, Shleifer, Summers, and Waldmann (1990), the inability of rational traders to ascertain the trades of the noise traders causes risk averse rational traders to avoid noise-prone securities. The classical paradigms of Glosten and Milgrom (1985) and Kyle (1985) also preclude rational agents from observing the orders of noise (or liquidity) traders. To understand why it may be difficult to predict irrational sentiment, suppose that among a hundred irrelevant information items that can potentially influence irrational investors, one turns out to be especially salient to these individuals. A rational trader who dismisses irrelevant information items may have trouble identifying which of the hundred items is the salient one that tempts the irrational traders into error.

back from prices to cash flows, the price effect of the irrational trades is not completely reversed, because the sentiment induced buying has a positive effect on fundamentals. As we show, when this feedback effect is sufficiently strong, the expected gains from early irrational trading outweighs the expected loss from late irrational trading, so that the combined profits can on average be positive and can even exceed the profits of traders with fundamental information.

Feedback allows early irrational investors to, in effect, exploit a kind of second order private information— not about fundamentals per se, but about the future order flow of investors with similar psychological biases. However, this argument does not require that the irrational investors be sophisticated enough to anticipate the feedback effect. Indeed, we assume that irrational traders ignore the reverse causality from prices to cash flows. The sequential arrival of correlated irrational trades automatically positions the early irrational investors to profit.

In our model, irrational trading causes prices to deviate from fundamental value. Nevertheless, market prices are efficient in the sense that, conditional on all available public information, prices follow a random walk. In equilibrium there are no profitable trading rules based upon public information. Thus, the model shows that irrational investor psychology can have real effects even if markets are, in a standard sense, efficient. Furthermore, the profitability of irrational trading suggests that these real effects may be persistent.

As indicated above, our analysis is closely related to some earlier models of securities trading. Subrahmanyam and Titman (2001) and Khanna and Sonti (2004) analyze feedback from the stock market to cash flows in a setting with fully rational investors. Here we examine the consequences of imperfectly rational trading. The relevance of having investors receive information at different times was previously explored by Froot, Scharfstein, and Stein (1992) and Hirshleifer, Subrahmanyam, and Titman (1994), both of which considered settings with rational investors and no feedback.[8]

[8]The main focus of these papers is on information and herding rather than the profitability of irrational trading. However, in the Froot, Scharfstein and Stein model, investors can potentially realize abnormal profits by trading on noise. In their model, the first investors who trade on the noise can profit from the price pressure of later-arriving investors who trade on the same noise. Noise investors

The remainder of the paper is structured as follows. Section 2 describes the economic setting. Section 3 derives asset demands, expected profits to different kinds of traders, and numerical comparisons of the profitability of different kinds of traders. Section 4 provides a simplified version of the model and analytic results about the profitability of different trader classes, while Section 5 concludes.

2 The Economic Setting

2.1 The Firm

Consider a firm that generates a single random terminal payoff. Equity claims on the firm are traded at each of dates 1 and 2, at prices P_1 and P_2. At date 3 the firm pays out the amount $F = \theta + \epsilon + \delta$, where θ and ϵ are independent normal random variables with mean zero. The random variable δ captures the effect of feedback. Specifically, we assume that the prices generated at dates 1 and 2 have an effect on cash flows generated at date 3. For example, the firm may be perceived as offering better employment opportunities if its stock price is rising, thus enabling the firm to attract more productive employees while offering the same wages.[9]

More specifically, we model this feedback through the strategies of the firms stakeholders, who choose how much to invest in their relationship with the firm. In particular, we assume that employees and suppliers choose how much to invest in specific human or physical capital, at least in part based on the expected profitability of its relation with the firm. For simplicity, we consider a single representative stakeholder (thereby abstracting from the free-rider problem among stakeholders in contributing to the firm). We assume that the stakeholder decides upon his or her investment at date 2 after observing market prices P_1 and P_2. We further assume that the marginal benefit to the

(early and late) on average make money if they can reverse their trades at a favorable price in the final trading round. The ability of noise investors to reverse at a favorable price derives from the assumption that all traders reverse their positions in the last round. Hirshleifer, Subrahmanyam, and Titman (1994) examine a setting that is similar except that the reversal of trades is endogenous. Owing to their risk aversion, investors wish to unwind their trades once their information (or noise) is revealed to the market. In this setting, which assumes no feedback, it is not possible to profit by trading on noise.

[9] The possibility that stock prices influence employees is considered in more detail in Subrahmanyam and Titman (2002).

stakeholder from the investment is increasing in the stakeholders expectation of θ (the basic, non-feedback component of the firms cash flow), $z = E(\theta|P_1, P_2)$.

Let X be the stakeholders investment. It is convenient to capture the effect of z on the marginal benefits to stakeholders of investing in its relation with the firm with a specific functional form. We specify the profit to stakeholders of investing X as

$$\Pi(X; z) = C_0 + C_1 z + K_0 X - \frac{K_1 X^2}{2z}, \tag{1}$$

where constants $C_0, C_1, K_0, K_1 > 0$.

Setting the derivative with respect to X equal to zero, the optimal investment is

$$X^*(z) = \left(\frac{K_0}{K_1}\right) z, \tag{2}$$

which is proportional to z. This solution describes the stakeholders investment indirectly as a function of the market prices P_1 and P_2 observed at date 2.

We further assume that the firm will be more profitable if its stakeholders invest in their relationship (e.g., profits are higher if the workers choose to be better trained). Specifically, the component of profitability δ is assumed to be proportional to the stakeholders investment,

$$\delta(X) = C_2 X. \tag{3}$$

So in equilibrium, profit is

$$\delta(X^*) = \left(\frac{C_2 K_0}{K_1}\right) z, \tag{4}$$

which is proportional to z.

Accounting for this feedback, the date 3 (terminal) cash flow generated by the firm can be expressed as

$$F = \theta + \epsilon + kE(\theta|P_1, P_2), \tag{5}$$

where

$$k = \frac{C_2 K_0}{K_1}. \tag{6}$$

2.2 The Investors

All traders behave competitively. We assume that there are two types of rational informed investors. The *early informed* learn precisely the realization of θ when the market opens at date 1, while the *late informed* do not learn the realization of θ until the market opens at date 2. The error term ϵ remains unknown at both trading dates. The presence of both early and late informed trading causes the market maker to condition his or her trades on order flow at both trading dates.

In addition, there is a group of utility-maximizing irrational traders who mistakenly believe that the security pays off $\eta+\epsilon$, where η is a random variable which is independent of all other exogenous random variables. Thus, η has no inherent relation to fundamentals, but endogenously becomes related because of the feedback effect. We assume that η is normally distributed with mean zero and variance v_η. The irrational traders observe the realization of η, but the rational traders do not. For simplicity, we assume that the irrational traders do not anticipate the feedback effect.

The mass (or measure) of the early irrational traders is denoted by M, while the total mass of early- and late-trading irrational investors is normalized to unity, so that the mass of late-trading irrational investors is $1 - M$. Similarly, the mass of early-informed traders is N, and their total mass also is normalized to unity. Informed and irrational investors have negative exponential utility over terminal wealth with a common absolute risk aversion coefficient R.

Liquidity demand shocks for the claim in amounts of z_1 and z_2 arrive at dates 1 and 2, respectively. These shocks are normally distributed with mean zero, and are independent of each other and of θ and ϵ. The common variance of the liquidity shocks is denoted as v_z. These shocks can also be interpreted as uncorrelated irrational trades, in contrast to the correlated trades emanating from agents who observe η sequentially.

There is also a group of risk neutral market makers, who possess no information about the fundamental value of the risky security. These agents represent a competitive fringe of risk neutral traders (e.g., floor brokers, scalpers, or institutions who monitor trading floor activities) who are willing to absorb the net demands of the other traders at competitive prices. Competitive risk neutral market makers are only concerned about order flow if there are some truly informed traders. So our assumption that there are informed traders is important.

We further assume that rational investors (whether uninformed market makers or informed traders) do not directly observe the trades of the irrational investors. This is important because otherwise, irrational trading would not lead to any informational confounding on the part of rational traders.

3 Analysis

3.1 The Demands of Investors

To derive the linear equilibria, we begin by postulating that the prices are linear functions of the private information variable θ and the current and past liquidity demand shocks, such that

$$P_1 = a_1\theta + a_2\eta + a_3 z_1 \tag{7}$$

$$P_2 = b_1\theta + b_2\eta + b_3 z_1 + b_4 z_2. \tag{8}$$

Let x_1 and x_2 represent the demands of the early informed (rational) investors at dates 1 and 2, respectively. Since the date 2 wealth is conditionally normally distributed, one can use the mean-variance framework and standard methodology to show that the optimal risky holdings of each early- and late-trading investor at the end of date 2 are identical and are

$$x_2 = \frac{\theta + kE(\theta|P_1, P_2) - P_2}{Rv_\epsilon}. \tag{9}$$

Let $E_r(P_2)$ and $v_r(P_2)$ denote the mean and variance of P_2 conditional on the information set of the early informed at date 1. This information set consists of θ and the market price P_1, so although the rational informed traders do not know η precisely, they infer it partially from market prices.[10] The appendix shows that the optimal date 1 demand

[10]The next section discusses a simplified case in which the informed are allowed to precisely observe the variable η with a lag. Similar results obtain in this alternative formulation.

of an early-informed trader is

$$x_1 = \left[\frac{E_r(P_2) - P_1}{R}\right]\left[\frac{1}{v_r(P_2)} + \frac{1}{v_\epsilon}\right] + \frac{\theta - E_r(P_2)/(1+k)}{Rv_\epsilon}. \tag{10}$$

The demand represented by (10) consists of two components, one to exploit the expected price appreciation across dates 1 and 2, and another to lock in at the current price the expected demand at date 2.

It can easily be shown that the date 1 demand of the late-trading informed investors equals zero in equilibrium. Intuitively, this occurs for two reasons. First, the equilibrium date 1 price does not offer a risk premium because of the presence of risk-neutral market makers. Second, the late-trading investors cannot hedge their date 2 demand in advance, because conditional on their date 1 information set (which does not contain the informational variable θ), the expected date 2 price is unbiased, so that their expected date 2 trade is zero.[11]

Let y_1 and y_2 represent the demands of the early irrational traders at dates 1 and 2. Again the date 2 demands of the early and late irrational traders are identical,

$$y_2 = \frac{\eta - P_2}{Rv_\epsilon}. \tag{11}$$

Let $E_n(P_2)$ and $v_n(P_2)$ denote the mean and variance of P_2 conditional on the information set of the early irrational traders at date 1 (this information set includes P_1 and their signal η). Analogous to (10), the date 1 demand of an early irrational trader is

$$y_1 = \left[\frac{E_n(P_2) - P_1}{R}\right]\left[\frac{1}{v_n(P_2)} + \frac{1}{v_\epsilon}\right] + \frac{\eta - E_n(P_2)}{Rv_\epsilon}, \tag{12}$$

whereas the date 1 demand of a late irrational trader equals zero for the same reason as for the late informed trader.[12]

3.2 Definition of Equilibrium

Our equilibrium concept closely parallels Vives (1995) and Hirshleifer, Subrahmanyam, and Titman (1994). Specifically, we assume that at dates 1 and 2, informed investors

[11] The proof of this intuitive assertion is available from the authors upon request.

[12] A proof is available upon request.

submit demand schedules ('limit orders') that are functions of their information and the market prices. The risk neutral market makers observe the combined demand schedules of the informed and liquidity traders and set competitive prices at each date. Let γ_1, γ_2 denote the aggregate demand schedules at dates $t = 1, 2$.

Because market makers are risk neutral and competitive, they set prices that are semi-strong form efficient. Thus, at each date the security's price is equal to the expectation of the terminal cash flow of the security, conditional on the information set of the market makers, i.e.,

$$\begin{aligned} P_1 &= E\left[F|\gamma_1(\cdot)\right] \\ P_2 &= E\left[F|\gamma_1(\cdot), \gamma_2(\cdot)\right]. \end{aligned} \tag{13}$$

The date 3 price, P_3, is equal to the final value of the claim, F. We will consider linear equilibria, wherein pricing functions are linear in the random variables θ, η, z_1, and z_2. Given such functions, it can easily be shown that the demand schedules can be written as

$$\begin{aligned} \gamma_1(P_1) &= f(\tau_1) + f_p(P_1) \\ \gamma_2(P_2) &= g(\tau_2) + g_p(P_2), \end{aligned} \tag{14}$$

where $f(\cdot)$, $f_p(\cdot)$, $g(\cdot)$, $g_p(\cdot)$ are non-stochastic linear functions, while τ_1 and τ_2 are linear combinations of the information variable θ, the irrational noise variable η, and the liquidity trades z_1 and z_2. The informative parts of the demand schedules are the variables τ_1 and τ_2. We therefore have

$$\begin{aligned} P_1 &= E\left[F|D_1(\cdot)\right] = E\left[F|\tau_1\right] \\ P_2 &= E\left[F|D_1(\cdot), D_2(\cdot)\right] = E\left[F|\tau_1, \tau_2\right]. \end{aligned} \tag{15}$$

3.3 Equilibrium Prices

Given the aggregate demands $\gamma_1 = Nx_1 + My_1 + z_1$ and $\gamma_2 = x_2 + y_2 + z_1 + z_2$, it can be shown that the components of the demand schedules which are informative about final value take the form:

$$\tau_2 = \theta + \eta + Rv_\epsilon(z_1 + z_2)$$

$$\tau_1 = N\left\{\left[\frac{E_r(P_2)}{R}\right]\left[\frac{1}{v_r(P_2)} + \frac{k}{(1+k)v_\epsilon}\right] + \frac{\theta}{Rv_\epsilon}\right\} + M\left[\frac{E_n(P_2)}{Rv_n(P_2)} + \frac{\eta}{Rv_\epsilon}\right] + z_1.$$

Given the expressions for the early and late irrational trader demands, it is easy to show that the expected profits of the late irrational traders are

$$\pi_{nl} = E[x_2(F - P_2)] = v_\theta\left[\frac{-b_1}{Rv_\epsilon}\left(1 - \frac{b_1}{1+k}\right)\right] + v_\eta\left[\frac{1-b_2}{Rv_\epsilon}\left(-\frac{b_2}{1+k}\right)\right] + v_z\frac{(b_3^2 + b_4^2)}{(1+k)Rv_\epsilon},$$

and the expected profits of the early irrational traders are

$$\begin{aligned}\pi_{ne} = E[x_2F - (x_2 - x_1)P_2 - x_1P_1] = R^{-1}v_\theta\left\{\frac{n_1}{v_n(P_2)} - a_1\left[\frac{1}{v_n(P_2)} + \frac{1}{v_\epsilon}\right]\right\}(b_1 - a_1)\\ +R^{-1}v_\eta\left\{\frac{n_2}{v_n(P_2)} - a_2\left[\frac{1}{v_n(P_2)} + \frac{1}{v_\epsilon}\right]\right\}(b_2 - a_2)\\ +R^{-1}v_z\left\{\frac{n_3}{v_n(P_2)} - a_3\left[\frac{1}{v_n(P_2)} + \frac{1}{v_\epsilon}\right]\right\}(b_3 - a_3) + \pi_{nl}.\end{aligned}$$

The above expressions indicate that the early irrational traders profit from feedback because the date 2 price is correlated both with their irrational assessment η as well as with the fundamental θ. Indeed, the stronger the feedback, the more the price moves at date 2.

The solution process for the equilibrium proceeds as follows. First, observe that $E_r(P_2) = E(P_2|P_1, \theta)$, and can be written as $r_1\theta + r_2\eta + r_3z_1$. Further, $E_n(P_2) = E(P_2|P_1, \eta)$ and can be written as $n_1\theta + n_2\eta + n_3z_1$. In addition, $v_r(P_2)$ and $v_n(P_2)$ are not functions of the realizations of the random variables, but are well-known functions of the price coefficients and variances of the random variables.[13] These facts allow us to solve for the equilibrium value of coefficients a_t and b_t, $t = 1, 2, 3$ in the price functions postulated in (8) and (7); details appear in the appendix. The following proposition describes properties of equilibrium price changes and the equilibrium expected profits of the rational and irrational traders.

Proposition 1 *In the general model:*

1. *Price changes are serially uncorrelated.*

[13]We assume that irrational traders accurately understand the coefficients of the market makers' pricing function. Our purpose is not realism, but modeling parsimony: we wish to show that a single irrationality can, when combined with feedback, generate excess profits for irrational traders.

2. *There exists a non-empty set of exogenous parameter values under which the ex ante expected profits of the irrational traders are positive and exceed those of the rational informed traders.*

Part 1 follows from the fact that market makers are risk-neutral and set prices to be expectations of final value conditional on all public information. Hence prices follow a martingale.

Part 2 of the proposition is demonstrated by means of numerical comparative statics on irrational trader profits with respect to changes in the strength of the feedback effect, which we summarize by k.[14] (see Figure 1), and on the expected profit differential between irrational and rational traders (See Figure 2).

In Figure 1, the late irrational expected profits are always negative. The early irrational expected profits are negative when $k = 0$, but increase as k increases. The ex ante total expected profits also start negative and increase with k.

Further numerical analysis (not reported) indicates that the threshold level of k above which the total irrational expected profits are positive is increasing in the ratio v_θ/v_η, the ratio of the variances of the inherent information, θ, and of the irrational belief, η, and decreasing in the variance of liquidity trades v_z; these are intuitive results. Increasing v_θ/v_η increases the strength of the price move against the late irrational traders. This adversely affects their expected profits, and consequently increases the threshold k. In contrast, increasing v_z, the variance of liquidity trading causes the market to be more liquid, which increases ex ante irrational trader profits.

We also find, somewhat surprisingly, that the threshold level of k is decreasing in the risk v_ϵ that is not resolved prior to the terminal date by the signal received by the rational informed. This occurs because increasing v_ϵ decreases the size of the position held by the late irrational traders, which mitigates their losses. Consequently, weaker feedback suffices for the ex ante expected profits of the irrational traders to be positive.

Figure 2 compares the total ex ante expected profits earned by irrational traders

[14] From equations (5) and (6), the sensitivity of cash flows to stock prices (i.e., the strength of the feedback effect) depends on k, which in turn is optimally determined by the parameters of the stakeholder profit function as in (2).

with the total ex ante expected profits earned by rational informed traders as a function of the feedback parameter. We graph the difference in profits relative to the informed profit (which is always positive) in the denominator. When the feedback parameter k is low, this difference is negative, indicating that irrational traders do worse than rational informed traders. However, irrational traders do better in relative terms as k increases, and earn higher profits than rational informed traders when k is greater than approximately 12.

3.4 Discussion

The findings illustrated in Figures 1 and 2 confirm the intuition offered in the introduction. When early irrational investors buy, market makers cannot determine whether the trade comes from an irrational investor or from an informed investor. The market maker thus responds by increasing the price, which through feedback increases the firm's fundamental value. The late irrational investors make similar trades, further increasing the price, which again increases the fundamentals because of feedback from price to cash flows. The later buy orders cannot be entirely anticipated by the market-maker, because there was a chance that the early buys came from intertemporally uncorrelated liquidity traders rather than intertemporally correlated irrational traders.

The late irrational investors on average lose money because they overpay even relative to the improved fundamentals. However, from an ex ante perspective, the expected profits of an irrational investor (who could turn out to trade either early or late) are positive. These irrational investors profit, when they trade early, by effectively trading on 'inside information' about the future order flow and, through feedback, its effect on fundamentals. If feedback is strong enough, this "pseudo-information" can be more valuable than the actual information acquired by the informed investors.

4 A Simplified Model

The general model considered in the last section provides intuitive numerical comparative statics but does not lend itself to analytical results. The difficulties in finding a closed-form solution are caused by the intertemporal hedging demands of the informed and irrational agents. As demonstrated by Equations (10) and (12), these demands depend non-linearly on both the date 1 and date 2 price coefficients, and this presents technical difficulties in solving the fixed-point problem inherent in the rational expectations equilibrium.

In the simplified model, the early irrational investors act as expected utility maximizers, but suboptimally act as one-period (myopic) agents and the informed investors receive information only at date 2. We do *not* require rational informed traders to behave myopically. Despite the fact that our assumptions require the early irrational trader to trade suboptimally (from his perspective) and allow rational traders to behave optimally, we will show that irrationality can still be ex ante profitable.

Each early irrational investor trades at date 1, and reverses his trade at date 2. Further, he forms his date 1 demand under the belief that the date 2 payoff is $\eta+\epsilon$. The total mass of early irrational traders is specified to equal M. As stated above, we assume that there is no informed trading at date 1, but at date 2 a unit mass of informed agents enter the market. In addition a mass $1-M$ of late irrational traders enter the market at date 2. Under these assumptions, it is easy to show that the irrational investor's trade equals

$$y_1 = \frac{\eta - P_1}{Rv_\epsilon}.$$

The date 2 trades of the rational and irrational informed agents as a function of the price remain unchanged in this framework.

The market makers observe the variables $\tau_1 \equiv M\eta + Rv_\epsilon z_1$ at date 1 and $\tau_2 \equiv \theta + (1-M)\eta + Rv_\epsilon(z_1+z_2)$. Now, the date 2 price is given by $P_2 = E(F|\tau_1, \tau_2) = (1+k)E(\theta|\tau_1, \tau_2)$. Let $E(\theta|\tau_1, \tau_2) = b_1'\theta + b_2'\eta + b_3'z_1 + b_4'z_2$. Then, using standard properties of normal distributions, the equilibrium values of the coefficients in this linear function are given as follows:

$$
\begin{aligned}
b'_1 &= v_\theta(M^2 v_\eta + R^2 v_\epsilon^2 v_z)/D & (16)\\
b'_2 &= R^2 v_\epsilon^2 v_\theta v_z (1-2M)/D & (17)\\
b'_3 &= -MRv_\epsilon v_\eta v_\theta (1-2M)/D & (18)\\
b'_4 &= Rv_\epsilon v_\theta (M^2 v_\eta + R^2 v_\epsilon^2 v_z)/D, & (19)
\end{aligned}
$$

where

$$D \equiv M^2 v_\eta (5R^2 v_\epsilon^2 v_z + v_\theta) - 4MR^2 v_\epsilon^2 v_\eta v_z + R^2 v_\epsilon^2 v_z (R^2 v_\epsilon^2 v_z + v_\eta + v_\theta) > 0. \quad (20)$$

The coefficients on η and z_1 in the date 1 price are zero. This again follows from the fact that the date 1 conditional expectation of market makers,

$$E(F|\tau_1) = E[\{\theta + kE(\theta|\tau_1, \tau_2)\}|\tau_1]$$

is equal to zero because $E(\theta|\tau_1) = 0$ and

$$E[E(\theta|\tau_1, \tau_2)]|\tau_1) = [Mv_\eta b'_2 + Rv_\epsilon v_z b'_3]\tau_1/\text{var}(\tau_1)$$

is also equal to zero (from (17) and (18)).

It is intuitively clear that the early irrational traders cannot profit unless the mass of late irrational traders is sufficiently large. It turns out that the expected profits of the early irrational traders are positive if and only if $M < 0.5$. For the remaining part of this section, we will explore in more detail the implications that arise when irrational traders earn expected profits, and will thus assume that the condition $0 < M < 0.5$ holds. Given this assumption, and the equilibrium price coefficients described above, it is straightforward to derive the following proposition.[15]

Proposition 2 *1. The irrational signal η, and the trades of the early irrational traders are both positively correlated with the fundamental value of the firm's claim so long as $k > 0$.*

[15] All propositions in this section are proved in the appendix.

2. *The correlation between the trades of the late irrational traders and the firm's date 3 fundamental value is of ambiguous sign. However, for M close to zero, this correlation is positive so long as $k > 0$ and v_η is sufficiently large.*

Proposition 2 indicates that irrational trading affects the firm's fundamental value. Indeed, the trades of the early irrational investors are positively correlated with the fundamental value so long as there is feedback from prices to cash flows. For small enough mass of the early irrational M, the correlation between the trades of the late irrational investors and the fundamental value is positive so long as the variance of the irrational signal is sufficiently large. This is intuitive; if there are not too many early irrational investors, then late irrational trades are of the same sign as those of the early irrational traders because there is not very much reversal of early irrational positions. So both are positively correlated with fundamentals.

The following proposition describes the correlation between irrational trades and price moves.

Proposition 3 1. *The trades of the early irrational traders are positively correlated with the date 2 price move, $P_2 - P_1$.*

2. *The trades of the late irrational traders are positively correlated with the date 2 price move, $P_2 - P_1$ if and only if*

$$v_\theta(1+k)(R^2 v_\epsilon^2 v_z + M^2 v_\eta) < R^2 v_\epsilon^2 v_\eta v_z (1-2M). \tag{21}$$

3. *The trades of the late irrational traders are positively correlated with the trades of the early irrational traders if and only if $1 + k > b_2^{\prime -1}$, i.e., if and only if*

$$\frac{D}{R^2(1+k)v_\epsilon^2 v_\theta v_z(1-2M)} > 1. \tag{22}$$

Part 1 indicates that irrational traders do indeed affect prices. Intuitively, market makers think that their trades might have come from informed investors. Together, parts 2 and 3 provide conditions under which the late irrational investors tend to trade in the same direction as early irrational investors, and in the same direction as the date 2 price move.

It is possible for the late irrational trades to be negatively correlated with both the early irrational traders and the date 2 price move. This happens when the inequalities in (21) and (22) are reversed. To see why, suppose that there is strong feedback, so that the rational informed buy very aggressively on a positive signal. In this case, late irrational investors may short stock even if their signal is positive because the price they face is high when there is strong informed buying.

We now describe autocorrelation patterns in prices and order flows.

Proposition 4 *1. Equilibrium order flows are positively autocorrelated.*

2. *Unconditional price changes are serially uncorrelated in equilibrium; i.e.,* $\text{cov}(P_3 - P_2, P_2 - P_1) = 0.$

3. *Equilibrium prices exhibit positive autocorrelation conditional on the irrational signal, and negative autocorrelation conditional on the rational signal. Specifically,*

$$\text{cov}(P_3 - P_2, P_2 - P_1|\eta) > 0$$

$$\text{cov}(P_3 - P_2, P_2 - P_1|\theta) < 0.$$

Part 1 indicates that order flows are serially dependent because of the sequential arrival of irrational traders. Part 2 again confirms that the results are not driven by market inefficiency; prices at each date are equal to the expected value of an equity claim on the firm conditional on all publicly available information. Together parts 1 and 2 show that serial dependence in order flow is not inconsistent with serial independence in price movements. These results are thus consistent with the positive autocorrelation in order imbalances but virtually zero autocorrelation in daily stock returns documented by Chordia, Roll, and Subrahmanyam (2002).

Part 3 of the above proposition documents that prices exhibit persistence after controlling for the irrational signal. This happens because order flows are noisy transformations of the rational investors' trades, so that prices underreact to the valid information signal θ. On the other hand, because the trades of the irrational investors get mixed in with those of the informed traders, prices overreact to irrational trades and consequently exhibit reversals after controlling for the rational signal.

The expected profits of the early irrational traders in terms of the price coefficients are

$$\pi_{ne} = E[x_1(P_2 - P_1)] = \frac{(1+k)b_2' v_\eta}{R v_\epsilon},$$

and those of the late irrational traders are

$$\pi_{nl} = E[x_2(F - P_2)]$$

$$= \frac{v_\theta[-b_1'(1+k)](1-b_1') + v_\eta[1-b_2'(1+k)](-b_2') + (1+k)v_z(b_3'^2 + b_4'^2)}{R v_\epsilon}.$$

The ex ante expected profits of the irrational traders are $\pi = M\pi_{ne} + (1-M)\pi_{nl}$. Substituting for the price coefficients, we find that the ex ante expected profits are

$$\pi = \frac{R v_\epsilon v_\eta v_\theta v_z (1-2M)[(k+2)M - 1]}{D}, \tag{23}$$

where D is as given in (20). This leads to the following proposition.

Proposition 5

1. *If there is no feedback, i.e., $k = 0$, then the ex ante expected profits of irrational traders are always negative.*

2. *The ex ante expected profits of irrational traders are positive so long as $k > (1 - 2M)/M$.*

Part 1 confirms that when there is no feedback, irrational trading is unprofitable. Part 2 indicates that, in contrast, sufficient feedback makes irrational trading profitable. The intuition is the same as that outlined in the preceding section. Again, the expected profits are not a consequence of sophisticated exploitation of the feedback effect; indeed irrational investors are naïve and simply ignore this effect. They inadvertently profit from feedback because their trades are correlated with those of later irrational investors.

The previous proposition also illustrates the critical role of the assumption that irrational investors do not all receive their signals at the same time. Recall that the profits of the early informed traders are positive if and only if $M < 1/2$. Thus, the ex ante expected profits of the irrational traders can be positive only if the mass of early informed agents is less than 0.5. Part 2 of Proposition 5 demonstrates that as the mass of early irrational traders, M, goes to zero, the bound on k goes to infinity. This indicates that for any finite level of feedback, a strictly positive mass of early irrational traders is necessary for the irrational traders to earn positive expected profits. Thus, for any finite k, M has to lie in the open interval (0, 0.5) for irrational traders to earn positive expected profits.

Further, the expected profits of the informed traders in terms of the coefficients b'_i are

$$v_\theta(1 - b'_1)^2 + v_\eta b'^2_2 + v_z(b'^2_3 + b'^2_4).$$

Again, substituting for the price coefficients, we find that expected informed profits, denoted by π_i are

$$\pi_i = \frac{R^2 v_\epsilon^2 v_\theta v_z [v_\eta\{M^2 + (2M - 1)^2\} + R^2 v_\epsilon^2 v_z]}{D}. \tag{24}$$

Comparing equations (23) and (24), we have the following proposition.

Proposition 6 *A sufficient condition for the ex ante expected profits of irrational traders to be greater than the ex ante expected profits of informed traders is that*

$$k > \frac{Rv_\epsilon\{v_\eta[(2M - 1)^2 + M^2] + R^2 v_\epsilon^2 v_z\} + v_\eta(2M - 1)^2}{Mv_\eta(1 - 2M)}. \tag{25}$$

Proposition 6 indicates that, consistent with the numerical results of the general model, a sufficiently high feedback parameter causes the expected profitability of irrational traders to be greater than that of rational informed traders.

A similar finding obtains in a simpler framework where irrational investors submit exogenous orders as in Kyle (1985), so long as the orders of some investors are intertemporally correlated. The ex ante expected profits of such investors will be positive so long as the feedback parameter k is sufficiently large. This result is fundamentally different from the usual finding in the microstructure literature (as in Kyle (1985)) that noise traders on average lose money. Details are available from the authors.

We next describe comparative statics on the expected profit differential between the irrational and the rationally informed traders.

Proposition 7 *1. The ex ante expected profit differential between irrational traders and informed traders is increasing in the feedback parameter, k.*

2. *The ex ante expected profit differential between irrational traders and informed traders is increasing in the variance of the signal observed by irrational traders, v_η, so long as*

$$k > \frac{[R^2 v_\epsilon^2 v_z + v_\theta(Rv_\epsilon + 1)][1 - 2M]}{M(R^2 v_\epsilon^2 v_z + v_\theta)}. \tag{26}$$

3. *The ex ante expected profit differential between irrational traders and informed traders is increasing in the variance of the signal observed by informed traders, v_θ, so long as*

$$k > \frac{v_\eta[(2M-1)^2(Rv_\epsilon + 1) + RM^2 v_\eta v_\epsilon] + R^3 v_\epsilon^3 v_z}{M v_\eta(1 - 2M)}. \tag{27}$$

Proposition 7 Part 1 indicates, consistent with intuition, that the performance of the irrational traders relative to the rational informed ones increases with the feedback parameter. Furthermore, (Part 2) the expected profits of the irrational traders relative to those of informed traders increase in the variance of the irrational signal. Intuitively, the coordinating signal and feedback drive trades and profits. Finally, as indicated by Part 3, the expected profit differential increases in the variance of information v_θ so long as the feedback effect is strong. This is because as the ex ante variance of private information increases, the signal to noise ratio in the net demand increases, which causes the market maker to adjust prices more to irrational (as well as other) trades, thereby strengthening the feedback resulting from irrational trades.

Our final proposition describes how irrational trading affects economic resource allocation.

Proposition 8 1. *The optimal stakeholder contribution $X^*(z)$ is positively correlated with the irrational signal η.*

2. *The ex ante volatility of stakeholder contributions is increasing in the variance of the irrational signal, v_η.*

The above proposition shows that irrational investor optimism increases the commitment of stakeholders to the firm, and that greater variability of investor sentiment causes greater variability in stakeholder inputs to the firm. Thus, feedback has real economic consequences.

5 Concluding Remarks

An efficient financial market is often defined as a market in which all publicly available information is 'fully reflected' in the prices of securities. This concept is important, in part, because of the link between the information conveyed by market prices and the allocation of resources.[16] In addition to the allocation of capital, the efforts of workers and other stakeholders may be allocated more efficiently when the actions of rational informed agents make financial market prices more efficient. Our model focuses on the flip side of the coin and examines how irrationality, operating through market prices, affects resource allocation. Our finding that the feedback from stock prices to resource allocation can cause irrational trading to be profitable suggests that the effects of irrational trading on resource allocation may be persistent.

Although irrational trading affects market prices and distorts resources, it does not necessarily make markets informationally inefficient in the conventional sense. Indeed, in our model stock prices follow a random walk, and investors who receive no private information (real or imagined) are unable to realize abnormal returns.[17] As a result, standard tests of market efficiency do not provide a gauge of the extent to which irrational investors distort market prices. Even if risk-adjusted returns are unpredictable, irrational trading can induce substantial deviations of prices from fundamentals, and can cause substantial shifts in resource allocation.

While we have analyzed feedback-related phenomena at the firm level, our analysis also applies to industries that become susceptible to waves of investor sentiment. Indeed, the turn-of-the-millenium tech boom was consistent with sentiment influencing stakeholders as well as investment within the internet sector. High stock prices encouraged executives and programmers to leave secure high-level positions to join internet

[16] See, e.g., Hayek (1945) and recent models by Fishman and Hagerty (1989), Khanna, Slezak, and Bradley (1994), and Subrahmanyam and Titman (1999).

[17] Thus, in our setting standard tests would not identify any market efficiency. Nevertheless, in our setting it is possible to detect the effects of irrational trading indirectly by examining cross-sectional relationships. In particular, the model suggests that irrational investors will be more active in sectors such as high tech, where feedback is likely to be strong because of the interdependence of firms within this sector. Weston (2001) estimates a microstructure model and finds evidence that noise traders are especially active in the technology-heavy Nasdaq market.

startups (see footnote 6), and also allowed internet companies to raise capital and increase investment rapidly.[18] Furthermore, owing to positive externalities between the investments of different dotcom firms, favorable investor sentiment may have fed back into higher long-run firm profitability.[19] During this period, individuals and firms were being educated about the benefits of shopping on the internet. The activities of startups like Amazon, e-Bay, and Yahoo helped develop a population of regular Web users, which probably contributed to the profitability of these firms.

At the aggregate macroeconomic level, the irrationality in our model can be viewed as Keynesian "animal spirits" in the stock market. Irrational optimism or pessimism can affect aggregate corporate profitability through the feedback effect, and business activity may respond in a confirming way to irrational swings in sentiment. Such effects can be magnified when there are positive investment externalities across firms, as in Shleifer (1986) and Cooper and John (1988). Our approach suggests that even when aggregate stock price movements are motivated by irrational beliefs, the investors who drive these price movements may be making profitable investments. This in turn suggests that animal spirits can have a continuing influence on stock prices, investment, and business cycles.

[18]Hendershott (2004) reports that dot-com firms invested $21 billion raised from equity investors. Similarly, in the late 1990s, the vast investments in optical cable bandwidth capacity by Worldcom and other telecommunications companies were, according to commentators, induced by the booming valuation of telecom stocks (see, e.g., Frieden (2003)).

[19]The failure of many internet firms makes the low profitability of the losers highly salient. However, Shleifer (2000) argues that irrational exuberance in internet stocks helped remedy what would otherwise have been sub-optimal development of the internet. Further, Hendershott (2004) reports that investment in dot-com firms led to $39 billion in value at end of 2001, for an annualized return-on-equity capital invested of 19%. He concludes that "...despite the distorted price signals and contrary to popular perception, wealth was created during the dot-com investment boom."

Appendix

Derivation of Equation (10): The wealth of the early-informed trader, denoted by W^E, is

$$W^E = x_2 F - (x_2 - x_1)P_2 - x_1 P_1.$$

Let $\mu \equiv \theta + kE(\theta|P_1, P_2)$. Substituting for x_2 from (9) into the expression for W^E, we have

$$\begin{aligned} W^E &= \frac{(\mu - P_2)}{Rv_\epsilon}(\mu + \epsilon) - \frac{(\mu - P_2)}{Rv_\epsilon}P_2 - x_1(P_1 - P_2) \\ &= \frac{(\mu - P_2)^2}{Rv_\epsilon} + \frac{(\mu - P_2)\epsilon}{Rv_\epsilon} - x_1(P_1 - P_2). \end{aligned} \tag{28}$$

Now from the formula for the characteristic function of a normal distribution, if $u \sim \mathcal{N}(\mu, \sigma^2)$, then $E(exp(\nu u)) = exp(\mu\nu + (1/2)\sigma^2\nu^2)$. In our case, setting $u = W^E$, $\nu = -R$, and using the fact that, from the perspective of the early informed, the only unknown at date 2 is the random variable ϵ, we have

$$E(-exp(-RW^E)|\phi_2) = -exp\{-R[x_1P_2 - x_1P_1 + (\mu - P_2)^2/(2Rv_\epsilon)]\}. \tag{29}$$

It follows that at date 1, the early-informed traders maximize the derived expected utility of their date 2 wealth

$$E[[-exp\{-R[x_1P_2 - x_1P_1 + (\mu - P_2)^2/(2Rv_\epsilon)]\}]|\phi_1]. \tag{30}$$

Now, (30) can be written as

$$-[2\pi v_r(P_2)]^{-\frac{1}{2}} \int_{-\infty}^{\infty} exp\left\{-R\left[x_1P_2 - x_1P_1 + \frac{(\mu - P_2)^2}{(2Rv_\epsilon)}\right] - \frac{1}{2}\frac{(P_2 - E_r(P_2))^2}{v_r(P_2)}\right\} d(P_2 - E_r(P_2)). \tag{31}$$

Completing squares, the expression within the exponential above can be written as

$$-\left[\frac{1}{2}w^2 s + hw + l\right], \tag{32}$$

where

$$\begin{aligned} w &= P_2 - E_r(P_2) \\ h &= Rx_1 - \tfrac{(\mu - E_r(P_2))}{v_\epsilon} \\ s &= \tfrac{1}{v_r(P_2)} + \tfrac{1}{v_\epsilon} \\ l &= Rx_1(E_r(P_2) - P_1) + \tfrac{(\mu - E_r(P_2))^2}{2v_\epsilon}. \end{aligned}$$

Define $u \equiv \sqrt{s}w + h/\sqrt{s}$. Then, expression (32) becomes $-(1/2)u^2 + (1/2)h^2/s - l$. The Jacobian of the transformation from w to u is $s^{-\frac{1}{2}}$, and thus the integral (31) becomes

$$-[2\pi v_r(P_2)s]^{-\frac{1}{2}} \int_{-\infty}^{\infty} exp\left(-\frac{1}{2}u^2 + \frac{1}{2}\frac{h^2}{s} - l\right) du$$

$$= -\frac{1}{[v_r(P_2)s]^{\frac{1}{2}}} exp\left(\frac{1}{2}\frac{h^2}{s} - l\right). \tag{33}$$

Solving for the optimal x_1 by maximizing the above objective, we obtain (10). An identical technique allows us to derive (12). ‖

Derivation of the Equilibrium Price Coefficients in(8) and (7): First, simple application of the formulas for the conditional moments of multivariate normal distributions allow us to obtain expressions for the coefficients r_1-r_3 and n_1-n_3 in terms of the price coefficients. Specifically, r_1, r_2, and r_3 are given by the coefficients of θ, η, and z_1 in the scalar quantity $A_r B_r^{-1} C_r'$, where

$$A_r \equiv [a_1 b_1 v_\theta + a_2 b_2 v_\eta + a_3 b_3 v_z, b_1 v_\theta],$$

$$B_r \equiv \left[\begin{array}{cc} a_1^2 v_\theta + a_2^2 v_\eta + a_3^2 v_z & a_1 v_\theta \\ a_1 v_\theta & v_\theta \end{array}\right],$$

and

$$C_r \equiv [a_1\theta + a_2\eta + a_3 z_1, \theta],$$

whereas $v_r(P_2)$ is given by $A_r B_r^{-1} A_r'$.

Similarly, the coefficients n_1, n_2, and n_3 are given by the coefficients of θ, η, and z_1 in $A_n B_n^{-1} C_n'$, where

$$A_n \equiv [a_1 b_1 v_\theta + a_2 b_2 v_\eta + a_3 b_3 v_z, b_2 v_\eta],$$

$$B_n \equiv \left[\begin{array}{cc} a_1^2 v_\theta + a_2^2 v_\eta + a_3^2 v_z & a_2 v_\eta \\ a_2 v_\eta & v_\eta \end{array}\right],$$

and

$$C_n \equiv [a_1\theta + a_2\eta + a_3 z_1, \eta],$$

while $v_n(P_2)$ is given by $A_n B_n^{-1} A_n'$.

Plugging for $E_r(P_2)$ and $E_n(P_2)$ into the expressions for τ_1 and τ_2, we have

$$\tau_1 = \theta\left[\frac{Nr_1}{R}\left\{\frac{1}{v_r(P_2)} + \frac{k}{(1+k)v_\epsilon}\right\} + \frac{N}{Rv_\epsilon} + \frac{Mn_1}{Rv_n(P_2)}\right]$$

$$+\eta\left[\frac{Nr_2}{R}\left\{\frac{1}{v_r(P_2)}+\frac{k}{(1+k)v_\epsilon}\right\}+\frac{Mn_2}{Rv_n(P_2)}+\frac{M}{Rv_\epsilon}\right]$$
$$+z_1\left[\frac{Nr_3}{R}\left\{\frac{1}{v_r(P_2)}+\frac{k}{(1+k)v_\epsilon}\right\}+\frac{Mn_3}{Rv_n(P_2)}+1\right],$$

which can be written as

$$\tau_1 = k_1\theta + k_2\eta + k_3 z_1.$$

We then have $E(\theta|\tau_1,\tau_2) = m_1\tau_1 + m_2\tau_2$. Since $P_2 = (1+k)E(\theta|\tau_1,\tau_2)$, by equating coefficients, we have

$$b_1 = (1+k)m_1k_1 + (1+k)m_2 \tag{34}$$
$$b_2 = (1+k)m_1k_2 + (1+k)m_2 \tag{35}$$
$$b_3 = (1+k)m_1k_3 + (1+k)Rv_\epsilon m_2 \tag{36}$$
$$b_4 = (1+k)m_2Rv_\epsilon. \tag{37}$$

We solve for P_1 as

$$P_1 = E(\theta|\tau_1) + kE[E(\theta|\tau_1,\tau_2)|\tau_1] = E(\theta|\tau_1) + \left(\frac{k}{1+k}\right)E(P_2|\tau_1).$$

Define $D_1 = k_1^2v_\theta + k_2^2v_\eta + k_3^2v_z$ and $D_2 = k_1b_1v_\theta + k_2b_2v_\eta + k_3b_2v_z$. Then it follows from a simple application of the projection theorem that

$$a_1 = \frac{k_1^2v_\theta + kk_1D_2/(1+k)}{D_1} \tag{38}$$
$$a_2 = \frac{k_1k_2v_\theta + kk_2D_2/(1+k)}{D_1} \tag{39}$$
$$a_3 = \frac{k_1k_3v_\theta + kk_3D_2/(1+k)}{D_1}. \tag{40}$$

The expressions (34)-(37) and (38)-(40) define a system of seven non-linear equations in the seven price coefficients a_1-a_3 and b_1-b_4. This completes the solution procedure for the price coefficients. ‖

Proof of Proposition 1: Part 1 follows from the fact that the sequence of prices P_1, P_2 and F form a martingale, increments to which are serially uncorrelated. Part 2 is demonstrated by direct calculation, as illustrated in Figures 1 and 2. ‖

Proof of Proposition 2: The covariance between the trades of the early-irrational investors and the terminal value is

$$\text{cov}[(\eta - P_1)/(Rv_\epsilon), \theta + kE(\theta|P_1, P_2)] = \frac{kb'_2 v_\eta}{Rv_\epsilon}.$$

Since, from (17), b'_2 is positive if and only if $M < 0.5$, the first part of the proposition follows.

To show Part 2, note that the covariance between the trades of the late irrational investors and the terminal value is

$$\text{cov}[(\eta - P_2)/(Rv_\epsilon), \theta + kE(\theta|P_1, P_2)].$$

This covariance can be expressed in terms of the price coefficients as

$$-b'_1(1+k)(1+ka'_1)v_\theta + b'_2[1-(1+k)b'_2]v_\eta - (1+k)(b'^2_3 + b'^2_4)v_z.$$

Substituting for the coefficients b'_1-b'_4 from (16)-(19), the covariance becomes

$$-v_\theta D^{-1}[k^2 v_\theta(M^2 v_\eta + R^2 v_\epsilon^2 v_z) + k\{2M^2 v_\eta v_\theta + 2MR^2 v_\epsilon^2 v_\eta v_z - R^2 v_\epsilon^2 v_z(v_\eta - 2v_\theta)\} + v_\theta(M^2 v_\eta + R^2 v_\epsilon^2 v_z].$$

As $M \to 0$, the expression in square brackets approaches

$$\frac{-v_\theta[k^2 v_\theta + k(2v_\theta - v_\eta) + v_\theta]}{R^2 v_\epsilon^2 v_z + v_\theta + v_\eta},$$

which is positive so long as v_η is sufficiently large. ||

Proof of Proposition 3: The covariance in the first part is

$$\text{cov}[(\eta - P_1)/(Rv_\epsilon), P_2 - P_1] = (1+k)b'_2 v_\eta/(Rv_\epsilon),$$

which is positive so long as $b'_2 > 0$. By (17), this is the case if and only if $M < 0.5$.

Similarly, the covariance in Part 2 is

$$\text{cov}[(\eta - P_2)/(Rv_\epsilon), P_2 - P_1],$$

which is positive if and only if

$$b'_2 v_\eta > (1+k)(a'^2_1 v_\theta + b'^2_2 v_\eta + (b'^2_3 + b'^2_4)v_z.$$

By (16)-(19), the above condition is true if and only if (21) holds.

Finally, the covariance in Part 3 is

$$\text{cov}[(\eta - P_2)/(Rv_\epsilon), (\eta - P_1)/(Rv_\epsilon)],$$

which is positive if and only if

$$1 > 1 - b_2'(1 + k),$$

so that (22) follows from (17). ‖

Proof of Proposition 4: To prove part 1, we observe that prices form a martingale (as in the first part of Proposition 1) or perform an explicit calculation. In particular,

$$\text{cov}(F - P_2, P_2 - P_1) = b_1'(1 - b_1')v_\theta - b_2'^2 v_\eta - (b_3'^2 + b_4'^2)v_z.$$

Substituting for the equilibrium values of b_1', b_2', b_3', and b_4', the first part follows.

To prove part 2, note that the date 1 order flow, denoted by Q_1, is $Q_1 = M\eta/(Rv_\epsilon) + z_1$, whereas the date 2 order flow, Q_2, is

$$Q_2 = (1 - M)(\eta - P_2)/(Rv_\epsilon) + (\theta - P_2)/(Rv_\epsilon) + z_2 - M\eta/(Rv_\epsilon).$$

Since $P_2 = (1 + k)(b_1'\theta + b_2'\eta + b_3' z_1 + b_4' z_2)$, we have

$$\text{cov}(Q_1, Q_2) = Mv_\eta[\{1-M\}\{1-(1+k)b_2'\}-(1+k)b_2'-M]+Rv_\epsilon v_z[-(1+k)(1-M)b_3'-(1+k)b_3'].$$

Substituting for b_2' and b_3' from (17)-(18), the above covariance reduces to $Mv_\eta(1-2M)$, and part 2 follows.

Now let us consider part 3. The standard formula for a conditional variance-covariance matrix of a normal vector $X_1 \sim \mathcal{N}(0, \Sigma_1)$ conditional on another normal vector $X_2 \sim \mathcal{N}(0, \Sigma_2)$ is

$$\text{var}(X_1|X_2) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21},$$

where Σ_{ij} represents the covariance matrix between X_i and X_j. Now, we have that $P_3 - P_2 = \theta - E(\theta|\tau_1, \tau_2)$ and $P_2 - P_1 = (1+k)E(\theta|\tau_1, \tau_2)$. Letting $X_1 = [P_3 - P_2, P_2 - P_1]$ and $X_2 = \eta$, we find that

$$\text{cov}(P_3 - P_2, P_2 - P_1|\eta) = \frac{R^4 v_\epsilon^4 v_\theta^2 v_z^2 (2M - 1)^2 (1 + k)}{D^2},$$

which is always positive. Similarly, letting $X_1 = [P_3 - P_2, P_2 - P_1]$ and $X_2 = \theta$, we find that

$$\text{cov}(P_3 - P_2, P_2 - P_1|\theta) = -[1+k][(b_2'^2 v_\eta + (b_3'^2 + b_4'^2)v_z],$$

which is always negative. ‖

Proof of Proposition 5: This proposition follows directly from an examination of the expression on the right-hand side of (23). ‖

Proof of Proposition 6: Since D is a common denominator in both (23) and (24), comparing the expected profits reduces to comparison of the numerators on the RHS of these equations. So the expected profits of the irrational traders exceed those of the rational traders if and only if

$$Rv_\epsilon v_\eta v_\theta v_z(1-2M)[(k+2)M-1] > R^2 v_\epsilon^2 v_\theta v_z[v_\eta\{(2M-1)^2 + M^2\} + R^2 v_\epsilon^2 v_z].$$

Straightforward algebra shows the equivalence of the above condition to (25), so long as $M < 0.5$. ‖

Proof of Proposition 7: From (23) and (24), the expression for the profit differential, denoted by $\Delta\pi$, is

$$\Delta\pi = \frac{-Rv_\epsilon v_\theta v_z\left[kMv_\eta(2M-1) + M^2 v_\eta(5Rv_\epsilon + 4) - 4Mv_\eta(Rv_\epsilon + 1) + R^3 v_\epsilon^3 v_z + v_\eta(Rv_\epsilon + 1)\right]}{D}. \tag{41}$$

The derivative of the above expression with respect to k is

$$\frac{MRv_\epsilon v_\eta v_\theta v_z(1-2M)}{D}.$$

This is positive if and only if $M < 0.5$.

Similarly, the derivative of the right-hand side of (41) with respect to v_η is

$$\frac{R^3 v_\epsilon^3 v_z^2 v_\theta(1-2M)[kM(R^2 v_\epsilon^2 v_z + v_\theta) - (R^2 v_\epsilon v_z + Rv_\epsilon v_\theta + v_\theta)(1-2M)}{D^2}.$$

This is positive so long as $M < 0.5$ and (26) is satisfied.

Finally, differentiating the RHS of (41) with respect to v_θ yields

$$R^3 v_\epsilon^3 v_z^2[v_\eta\{(2M-1)^2+M^2\}+R^2 v_\epsilon^2 v_z][kMv_\eta(1-2M)-\{v_\eta(2M-1)^2(Rv_\epsilon+1)+M^2 v_\eta Rv_\epsilon\}-R^3 v_\epsilon^3 v_z,$$

which is positive so long as $M < 0.5$ and (27) holds. ‖

Proof of Proposition 8: Substituting for $X^*(z)$ from Equation (2) into Equation (1) indicates that the optimized stakeholder profit is given by

$$C_0 + \left(C_1 + \frac{K_0^2}{2K_1} \right) z$$

and the stakeholder investment is linear in $z \equiv E(\theta|P_1, P_2)$. The correlation between η and z is positive if and only if b_2' is positive, and this is true if and only if $M < 0.5$ (see equation (17)), proving Part 1 of the proposition.

It can be shown from (16)-(19) that the ex ante variance of z is

$$D^{-1} \left[v_\theta^2 (M^2 v_\eta + R^2 v_\epsilon^2 v_z \right] .$$

The derivative of this quantity with respect to v_η is

$$-D^{-2} \left[R^4 v_\epsilon^4 v_\theta^2 v_z^2 (2M - 1)^2 \right] ,$$

which is always negative, proving Part 2. ‖

References

Baker, M. P., J. C. Stein, and J. Wurgler, 2003, When does the market matter? stock prices and the investment of equity-dependent firms, *Quarterly Journal of Economics* **118**, 969–1005.

Barberis, N. and R. Thaler, 2003,. A survey of behavioral finance, . In G. Constantinides, M. Harris, and R. Stulz, eds., *Handbook of the Economics of Finance*, Chapter 18, pp. 1053–1123. (Elsevier Science Ltd, North-Holland, Amsterdam).

Blume, L. and D. Easley, 1990, Evolution and market behavior, *Journal of Economic Theory* **58**, 9–40.

Chordia, T., R. Roll, and A. Subrahmanyam, 2002, Order imbalance, liquidity, and market returns, *Journal of Financial Economics* **65**, 111–130.

Cooper, R. and A. John, 1988, Coordinating coordination failures in keynesian models, *Quarterly Journal of Economics,* **103**, 441–463.

DeLong, J. B., A. Shleifer, L. Summers, and R. J. Waldmann, 1990, Noise trader risk in financial markets, *Journal of Political Economy* **98**, 703–738.

DeLong, J. B., A. Shleifer, L. Summers, and R. J. Waldmann, 1991, The survival of noise traders in financial markets, *Journal of Business* **64**, 1–20.

Dong, M., D. Hirshleifer, S. Richardson, and S. H. Teoh, 2003, Does investor misvaluation drive the takeover market?, Working paper, Ohio State University, York University and Wharton School.

Fischer, P. and R. Verrecchia, 1999, Public information and heuristic trade, *Journal of Accounting and Economics* **27**, 89–124.

Fishman, M. and K. Hagerty, 1989, Disclosure decisions by firms and the competition for price efficiency, *Journal of Finance* **44**, 633–646.

Frieden, R., 2003, Fear and loathing in information and telecommunications industries: Reasons for and solutions to the current financial meltdown and regulatory quagmire, *The International Journal on Media Management* **5**, 25–38.

Froot, K., D. Scharfstein, and J. C. Stein, 1992, Herd on the street: Informational inefficiencies in a market with short-term speculation, *Journal of Finance* **47**, 1461–1484.

Glosten, L. R. and P. R. Milgrom, 1985, Bid, ask and transaction prices in a specialist market with heterogeneously informed traders, *Journal of Financial Economics* **14**, 71–100.

Hayek, F. v., 1945, The use of knowledge in society, *American Economic Review* **35**, 519–530.

Hendershott, R. J., 2004, Net value: Wealth creation (and destruction) during the internet boom, *Journal of Corporate Finance* **10**, 281–299.

Hirshleifer, D. and G. Y. Luo, 2001, On the survival of overconfident traders in a competitive security market, *Journal of Financial Markets* **4**, 73–84.

Hirshleifer, D., A. Subrahmanyam, and S. Titman, 1994, Security analysis and trading patterns when some investors receive information before others, *Journal of Finance* **49**, 1665–1698.

Hirshleifer, D., 2001, Investor psychology and asset pricing, *Journal of Finance* **64**, 1533–1597.

Ho, T. and R. Michaely, 1988, Information quality and market efficiency, *Journal of Financial and Quantitative Analysis* **5**, 357–386.

Huberman, G. and T. Regev, 2001, Contagious speculation and a cure for cancer, *Journal of Finance* **56**, 387–396.

Khanna, N., S. L. Slezak, and M. H. Bradley, 1994, Insider trading, outside search, and resource allocation: Why firms and society may disagree on insider trading restrictions, *Review of Financial Studies* **7**, 575–608.

Khanna, N. and R. Sonti, 2004, Value creating stock manipulation: Feedback effect of stock prices on firm value, *Journal of Financial Markets* **7**, 237–270.

Kogan, L., S. Ross, J. Wang, and M. Westerfield, 2003, The survival and price impact of irrational traders, NBER Working Paper 9434.

Kyle, A. S., 1985, Continuous auctions and insider trading, *Econometrica* **53**, 1315–1335.

Kyle, A. and F. A. Wang, 1997, Speculation duopoly with agreement to disagree: Can overconfidence survive the market test?, *Journal of Finance* **52**, 2073–2090.

Ofek, E. and M. Richardson, 2003, DotCom mania: The rise and fall of internet stock prices, *Journal of Finance* **58**, 1113–1137.

Polk, C. and P. Sapienza, 2003, The real effects of investor sentiment, Working paper, Northwestern University.

Rashes, M. S., 2001, Massively confused investors making conspicuously ignorant choices (MCI - MCIC), *Journal of Finance* **56**, 1911–1928.

Shiller, R. J., 2000, *Irrational exuberance* (Princeton University Press, Princeton, N.J.).

Shleifer, A., 1986, Implementation cycles, *Journal of Political Economy* **94**, 1163–1190.

Shleifer, A., 2000, Are markets efficient? No, arbitrage is inherently risky, *The Wall Street Journal*, A10.

Soros, G., 2003, *The Alchemy of Finance* (John, Wiley and Sons, Inc, Hoboken, NJ).

Subrahmanyam, A. and S. Titman, 1999, The going-public decision and the development of financial markets, *Journal of Finance* **54**, 1045–1082.

Subrahmanyam, A. and S. Titman, 2001, Feedback from stock prices to cash flows, *Journal of Finance* **56**, 2389–2413.

Vives, X., 1995, Short term investment and the informational efficiency of the market, *Review of Financial Studies* **8**, 125–160.

Wang, F. A., 1998, Strategic trading, asymmetric information and heterogeneous prior beliefs, *Journal of Financial Markets* **1**, 321–352.

Weston, J. P., 2001, Information, liquidity, and noise, working paper, Jones School of Management, Rice University.

Figure 1
Ex ante expected noise trader profits vs. feedback parameter
[var(theta)=.4,var(eta)=1.9,R=0.18,var(z)=.6,var(epsilon)=4.2,M=0.05,N=.12]

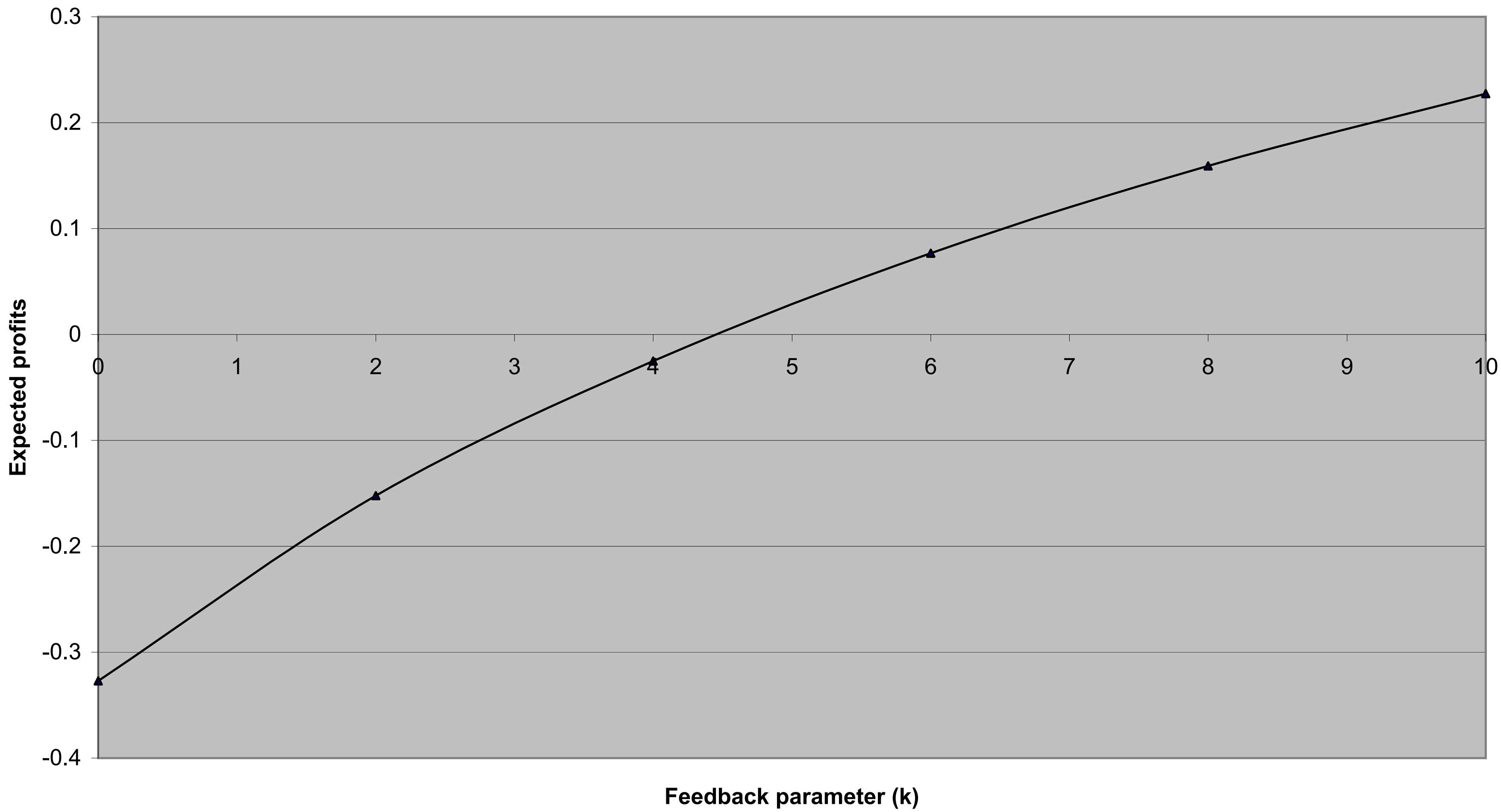

Figure 2
Difference between expected profits of irrational and informed traders, as a proportion of the expected profits of informed traders
[var(theta)=0.1, var(eta)=3.14, R=0.21, var(z)=0.99, var(epsilon)=3.16, M=0.08, N=0.06]